Lecture Notes in Mathematics 2018

Bei Hu

Blow-up Theories for Semilinear Parabolic Equations

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To Yi Cheng 程怡

Preface

I am grateful to Professor Zhengce Zhang (张政策) and his students from Xi'an Jiaotong university for correcting many typographic errors in my 2005 version of the lecture notes from their graduate reading seminar in the past few years. I am also grateful to the reviewers who offered constructive comments and suggested improvements of the manuscript. I would like to thank the editors of the LNM series for their suggestions, which improved the presentation of the materials and made the lecture notes more friendly to students. I would also like to thank Timothy McCoy for reading through the entire manuscript.

Notre Dame, IN *Bei Hu*
Fall 2010

These are the lecture notes of a course given at Xi'an Jiaotong university in summer of 2005. They are intended for beginning graduate students who have finished a first-year graduate course in basic partial differential equations.

The prerequisites include an understanding of the basic theory of the second-order equations such as

1. Maximum principles, basic existence and uniqueness theorems.
2. A priori estimates such as the Schauder estimates, the L^p estimates, the De Giorgi–Nash–Moser estimates, the Krylov–Safanov estimates.
3. The fixed point theorems.

There is an enormous amount of work in the literature about the blow-up behavior of evolution equations. It is our intention to introduce the theory by emphasizing the methods while avoiding the massive technical computations if possible. To reach this goal, we use the simplest equation to illustrate the methods; these methods very often apply to more general equations. No attempt is made during the lectures to include the most general theories and results.

I would like to use this opportunity to thank my colleagues at Xi'an Jiaotong University for their hospitality. In particular, I am grateful to Professor Jianzhong Shen (申建中), who corrected quite a few errors in an earlier version of the lecture notes. I am also grateful to Professor Lihe Wang (王立河), who invited me to give the lectures.

Xi'an, China *Bei Hu*
Summer 2005

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Chapter 1
Introduction

Blow-up phenomenon occurs in various types of nonlinear evolution equations. For example, they occur in Schrödinger equations, hyperbolic equations (see the papers Kalantarov–Ladyzhenskaya [81] and Deng [25, 26], Galaktionov–Pohozaev [54]), as well as in parabolic equations.

There are so many different applications and here we only list several important examples.

Example I. Heat equation with a source.
For the heat equation

$$u_t = \Delta u + f(x, t, u, \nabla_x u), \tag{1.1}$$

the variable u can be viewed as the temperature in a chemical reaction. A positive f represents a heat source, and the second-order derivative represents the diffusion.

In a situation where higher temperature will accelerate the chemical reaction and the chemical reaction will generate heat, what will happen? Unless the heat energy dissipates through diffusion, the temperature will likely become *very high*.

In solid fuel ignition [15, (1.28)], f may take the form e^u. The nonlinear terms $|u|^p$ or $|u|^{p-1}u$ are the most frequently studied examples. Even in the simplest form that f depends only on u and is nonnegative, there is a competition between the diffusion and the heat source, and it is not clear whether the temperature will become unbounded in finite time. This is physically interpreted as a dramatic increase of the temperature which leads to ignition. Some natural questions are

- Will the blow-up occur in finite time?
- If the blow-up occurs in finite time, where are the blow-up points? Can the blow-up occur in a region?
- What is the asymptotic behavior of the solution near the blow-up time?

These questions have been studied extensively in the literature.

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Example II. Nonlinear Schrödinger equation.
For the Schrödinger equation

$$iu_t + \Delta u + f(|u|)u = 0, \tag{1.2}$$

the variable u is a complex valued wave function and f is a real valued function taking real variables. In optics, the nonlinear Schrödinger equation describes wave propagation in fiber optics. In this case, the function u represents a wave and the equation describes the propagation of the wave through a nonlinear medium. The second-order derivative represents the dispersion; the term involving f represents the nonlinearity effects in a fiber. The nonlinear Schrödinger equation is also used to describe water waves.

There are numerous research papers in the literature on this equation. In the case $f(s) = s^{p-1}$, the solution may or may not blow up in finite time, depending on the exponent p and the magnitude of the initial data. Similar questions as in Example I (Will the blow-up occur? What are the possible blow-up locations? What is the asymptotic behavior?) are of interest.

Example III. Wave equation.
An example of the nonlinear wave equations may take the form

$$u_{tt} + ku_t = \Delta u + f(u), \tag{1.3}$$

where the Laplacian represents the dispersion. The term involving k represents the damping effect. The nonlinear function f represents the force and may take the form $|u|^p$. Depending on the range of p and the initial data, the solution may or may not blow up in finite time. The blow-up questions for this type of nonlinear wave equations have also been studied extensively.

Blow-up behavior for various evolution equations and systems have also been studied in numerous other papers. The survey papers Levine [86], Fila–Filo [34], Bandle–Brunner [10] and Deng–Levine [28] are the good starting point to get a sense of the current development of the blow-up theories.

There are several books (e.g., Samarskii–Galaktionov–Kurdyumov–Mikhailov [125], Galaktionov [51], and Quittner–Souplet [123]) with very detailed and extensive study on blow-up theory. These are very nice reference books for experts and for researchers who want to quote results in this field. Improperly posed problems were studied in Payne [114]. The goal of these lecture notes is to provide a one-summer-semester of material for beginning graduate students who have had 1 year of graduate level PDE theory. We choose only a very small subset of material from Example I, and emphasize the method rather than the generality. We have chosen the simple models to illustrate the method so that the students learn the method rather than the complicated and sometimes massive computations. Nevertheless, these methods very often apply to more general equations.

Consider the ordinary differential equation (ODE)

$$u_t = f(u), \tag{1.4}$$

where f is assumed to be Lipschitz continuous. This equation can be solved explicitly.

If $f(u(t_1)) = 0$, then by uniqueness, $u(t) \equiv u(t_1)$ for all $t \in (-\infty, \infty)$. Thus a non-constant solution cannot touch the zeros of the function f. Therefore blow-up can only occur if $u(t_1)$ starts at a point beyond all zeros of f. Assuming that this is the case (i.e., $f(s) > 0$ for $s \geqslant u(t_1)$), then

$$\int_{u(t_1)}^{u(t)} \frac{du}{f(u)} = t - t_1. \tag{1.5}$$

The right-hand side of the above equation is always bounded for finite t. It is clear that in this case, a necessary and sufficient condition for solutions to blow up is

$$\int^{\infty} \frac{du}{f(u)} < \infty; \tag{1.6}$$

this is the Osgood criterion, first introduced by Osgood in 1898 [113]. In the case $f(u) = -au + |u|^p$, $(a > 0)$. The solution blows up in finite time if and only if $p > 1$ and $u(0) > a^{1/(p-1)}$. In the case diffusion is included with zero Dirichlet boundary condition, the ODE solution can be used as an upper bound and therefore (1.6) is a necessary condition for blow-up to occur.

The *energy* associated with $f(u) = -au + |u|^p,\ (a > 0,\ p > 1)$ is

$$V(u) = \frac{a}{2}u^2 - \frac{1}{p+1}|u|^p u.$$

$V(-\infty) = +\infty$, $V(+\infty) = -\infty$, and there are only two local extrema: a local minimum at $u = 0$ with $V(0) = 0$, and a local maximum at $u = a^{1/(p-1)}$ with $V(a^{1/(p-1)}) = \frac{p-1}{2(p+1)}a^{(p+1)/p-1)}$. It is clear that

$$\int_0^t u_t^2(s)ds + V(u(t))) = V(u(0)).$$

Any solution starting in the interval $(-\infty, a^{1/(p-1)})$ converges to 0 as $t \to \infty$, and as mentioned above, any solution starting in the interval $(a^{1/(p-1)}, +\infty)$ blows up in finite time. In this case, the graph over the interval $(-\infty, a^{1/(p-1)})$ forms a well that attracts all solutions starting inside this interval.

The above argument is associated with a powerful tool of interest: the potential well theory (c.f. Payne–Sattinger [115], for hyperbolic equations). Instead of a dynamical system, it utilizes a functional in an appropriate Sobolev space.

A potential well is a region near a locally minimal potential energy. Solutions starting inside the well are global in time, and the energy is nonincreasing in time. Solutions starting outside the well and at an unstable point blow up in finite time. This theory has developed for a variety of nonlinear evolution equations.

We now return to the case of (1.1). When f is like $|u|^p$, the question of whether the solution blows up in finite time will depend on which of the following wins the competition: the nonlinear heat generation, or the heat dissipation by diffusion. If enough heat energy is dissipated (e.g., through the boundary), then blow-up may be prevented. It it not surprising that some finite time blow-up results require large enough initial data (initial heat energy). There are many results since the 1960s, and in these lecture notes, we shall cover selected topics in the field.

These lecture notes are organized as follows.

At the request of students, we list the preliminary material from elliptic and parabolic equations in Chaps. 2–4. Some exercises are included in these chapters to help the understanding of the material in the later chapters.

Chapter 5 is devoted to give answers to the question of whether blow-up will occur. We include the simple eigenvalue method from Kaplan [80] and Fujita [46]. We also cover the concavity method from Levine–Payne [90, 91] and Levine [85]. We also include some simple comparison methods. Some early papers also appeared in the 1970s, e.g., Tsutsumi [131], Hayakawa [70], Walter [133], Ball [7], Kobayashi–Siaro–Tanaka [82], Aronson–Weinberger [6]. There has been a parade of work since then for many similar problems arising from chemical reactions, fluid mechanics, turbulent flows, etc. Near the end of Chap. 5 we included a brief remark by a variety of mathematicians on extensions to other problems.

The study of blow-up behavior is dependent upon the behaviors of steady state solutions. Fortunately, when the equation for the steady state solution is simple enough, the solutions can be classified. In Chap. 6, we introduce the moving plane method from Gidas–Ni–Nirenberg [61]. For the equation $\Delta u + u^p = 0$, this method leads to the nonexistence of positive solutions for $p < \frac{n+2}{n-2}$ and a classification of the solutions to the problem $\Delta u + u^p = 0$ for $p = \frac{n+2}{n-2}$. The classification is actually related to the study of the Yamabe problem.

Chapter 7 is devoted to giving answers to the questions on the blow-up rate. If the order of blow-up is the same (but with possibly different constants) when compared to the ODE blow-up rate, then it is referred to as a type I blow-up rate. The method of using auxiliary functions and the comparison principle is a powerful one that has been successfully used by many authors; we choose the special method given by Friedman–McLeod [42], who did this to successfully obtain a blow-up rate. The scaling method is another powerful tool for studying many properties (e.g., regularity) of solutions to elliptic and parabolic equations. It is not surprising that this method is also very useful to obtain blow-up rates. The scaling method [73, 75] applied to blow-up rate estimate is also included in this chapter.

Asymptotically self-similar blow-up solutions are studied in Chap. 8. In a sense, solutions near the blow-up point can be characterized in much finer detail using self-similar solutions. The material in the first two sections is from Giga–Kohn [64, 65], while the last section deals with the construction of a general Lyapunov function in the one-space-dimensional case (Galaktionov [50], Zelenyak [148]).

The one-space-dimensional case is easier to deal with when compared to higher dimensional cases. With one space dimension and the time variable, the total dimension for the problem is two. In this case, a continuous curve in the x–t plane starting in the left half of the plane $\{(x,t);\ x<0\}$ cannot end up at the right half of the plane $\{(x,t);\ x>0\}$ without crossing the t-axis $\{x=0\}$. This argument *cannot* be extended to higher space dimensional cases. The special geometry makes it easier to count the number of zeros and makes intersection comparison possible. There are very nice books for one space variable case (or radially symmetric multi-space variable case) Samarskii–Galaktionov–Kurdyumov–Mikhailov [125] and Galaktionov [51]. Chapter 9 is meant to be a sampler of this material.

Chapter 2
A Review of Elliptic Theories

In Chaps. 2–4 we shall review material from first year PDE courses. These theories can be found in many books such as Chen–Wu [21], Chen [20], Gilbarg–Trudinger [67], Lieberman [94]. Compared to others, the books [21] and [20] are not intimidating even for beginners and therefore are excellent textbooks for beginning graduate students. The book [21] received the textbook excellence award from the education department of China. Some of the more classical theories can also be found in Ladyzenskaja–Solonnikov–Ural'ceva [83], Ladyzenskaja–Ural'ceva [84] and Friedman [43], and the classical maximum principles in Protter–Weinberger [118]. In this chapter we list the elliptic theories that we will need later on.

2.1 Weak Solutions and a Weak Maximum Principle

Let Ω be a domain in $\mathbb{R}^n$. Consider elliptic equations of divergence form in Ω:

$$Lu = -\sum_{i,j} D_j(a^{ij}D_iu + d^ju) + \sum_i (b^iD_iu + cu) = f + \sum_i D_if^i, \qquad (2.1)$$

where $D_i = \dfrac{\partial}{\partial x_i}$. To simplify notation, we shall use the summation convention where repeated index means summation. We can then drop the $\sum$ in the above equation. We assume that there exist positive constants λ, Λ such that

$$\lambda|\xi|^2 \leqslant a^{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2, \qquad \forall \xi \in \mathbb{R}^n, x \in \Omega, \qquad (2.2)$$

$$\sum_{i=1}^n \|b^i\|_{L^n(\Omega)} + \sum_{i=1}^n \|d^i\|_{L^n(\Omega)} + \|c\|_{L^{n/2}(\Omega)} \leqslant \Lambda. \qquad (2.3)$$

For $u, v \in H^1(\Omega)$, we set

$$a(u,v) = \int_\Omega \Big\{(a^{ij}D_iu + d^ju)D_jv + (b^iD_iu + cu)v\Big\}dx.$$

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Definition 2.1. For $f \in L^2(\Omega), f^i \in L^2(\Omega)$ and $g \in H^1(\Omega)$, we say that $u \in H^1(\Omega)$ is a *weak solution* of the Dirichlet problem

$$\begin{cases} Lu = f + \sum_i D_i f^i \text{ in } \Omega, \\ u = g \qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

if u satisfies

$$\begin{cases} a(u,v) = \langle f, v\rangle - \sum_i \langle f^i, D_i v\rangle, \quad \forall v \in H_0^1(\Omega), \\ u - g \in H_0^1(\Omega), \end{cases} \tag{2.5}$$

here we use the notation $\langle f, g\rangle = \int_\Omega fg dx$.

Definition 2.2. $u \in H^1(\Omega)$ is said to be a *weak subsolution* (*weak supersolution*), if

$$a(u,\phi) \leqslant (\geqslant)\ \langle f,\phi\rangle - \langle f^i, D_i\phi\rangle, \quad \forall \phi \in C_0^\infty(\Omega), \phi \geqslant 0. \tag{2.6}$$

Theorem 2.1 (Weak maximum principle). [21, p. 9, Theorem 4.2]. *Let the assumptions* (2.2), (2.3) *be in force, and*

$$\int_\Omega (c\phi + d^i D_i\phi)dx \geqslant 0, \ \ \forall \phi \in C_0^\infty(\Omega), \phi \geqslant 0.$$

If $u \in H^1(\Omega)$ is a weak subsolution, then for any $p > n$, we have

$$\operatorname*{ess\,sup}_\Omega u \leqslant \sup_{\partial\Omega} u^+ + C(\|f\|_{L^{np/(n+p)}(\Omega)} + \sum_i \|f^i\|_{L^p(\Omega)})|\Omega|^{(1/n)-(1/p)},$$

where C depends only on $n, p, \lambda, \Lambda, \Omega$ and b^i, d^i, c, and it is independent of the lower bound of $|\Omega|$.

2.2 Schauder Theory

The $C^{2+\alpha}$ Schauder theory for the Dirichlet problem of second order elliptic equations was established as early as the 1930s. The theory is complete as far as classical solutions are concerned.

Let Ω be a bounded domain. Consider the linear second order elliptic equation of non-divergence form:

$$Lu = -a^{ij}D_{ij}u + b^iD_iu + cu = f \quad \text{ in } \Omega. \tag{2.7}$$

Assume that there exist $\Lambda \geqslant \lambda > 0$ such that

$$\lambda|\xi|^2 \leqslant a^{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n, \tag{2.8}$$

$a^{ij}, b^i, c \in C^\alpha(\overline{\Omega})$ $(0 < \alpha < 1)$ and

$$\frac{1}{\lambda}\left\{ \sum_{i,j} |a^{ij}|_{\alpha;\Omega} + \sum_i |b^i|_{\alpha;\Omega} + |c|_{\alpha;\Omega} \right\} \leqslant \Lambda_\alpha. \tag{2.9}$$

Theorem 2.2 (Interior Schauder estimates). [21, p. 28, Theorem 4.3]. *Suppose that the coefficients in (2.7) satisfy the assumptions (2.8) and (2.9). Let* $u \in C^{2,\alpha}(\Omega)$ $(0 < \alpha < 1)$ *be a solution of (2.7). Then for* $\Omega' \subset\subset \Omega$, *we have*

$$|u|_{2,\alpha;\Omega'} \leqslant C\left(\frac{1}{\lambda}|f|_{\alpha;\Omega} + |u|_{0;\Omega}\right), \tag{2.10}$$

where C *depends only on* $n, \alpha, \Lambda/\lambda, \Lambda_\alpha$ *and* $\mathrm{dist}\{\Omega', \partial\Omega\}$.

Theorem 2.3 (Global Schauder estimates, Dirichlet). [21, p. 31, Theorem 5.3]. *Let the assumptions (2.8), (2.9) be in force, and* $\partial\Omega \in C^{2,\alpha}$ $(0 < \alpha < 1)$. *Suppose that* $u \in C^{2,\alpha}(\overline{\Omega})$ *is a solution of (2.7) satisfying the boundary condition* $u\Big|_{\partial\Omega} = g$ *for some* $g \in C^{2,\alpha}(\overline{\Omega})$. *Then*

$$|u|_{2,\alpha;\Omega} \leqslant C\left(\frac{1}{\lambda}|f|_{\alpha;\Omega} + |u|_{0;\Omega} + |g|_{2,\alpha;\Omega}\right), \tag{2.11}$$

where C *depends only on* $n, \alpha, \Lambda/\lambda, \Lambda_\alpha$ *and* Ω.

Theorem 2.4 (Global Schauder estimates, Neumann). [67, p. 127, Theorem 6.30]. *Let the assumptions (2.8), (2.9) be in force, and* $\partial\Omega \in C^{2,\alpha}$ $(0 < \alpha < 1)$. *Suppose that* $u \in C^{2,\alpha}(\overline{\Omega})$ *is a solution of (2.7) satisfying the boundary condition* $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g$ *for some* $g \in C^{1,\alpha}(\overline{\Omega})$, *where* n *is the unit exterior normal vector. Then*

$$|u|_{2,\alpha;\Omega} \leqslant C\left(\frac{1}{\lambda}|f|_{\alpha;\Omega} + |u|_{0;\Omega} + |g|_{1,\alpha;\Omega}\right), \tag{2.12}$$

where C *depends only on* $n, \alpha, \Lambda/\lambda, \Lambda_\alpha$ *and* Ω.

Remark 2.1. The term $|u|_{0;\Omega}$ can be dropped in (2.11), (2.12) if we further assume $c \geqslant 0$.

Remark 2.2. The estimates also extends to other oblique boundary conditions.

Consider the Dirichlet problem

$$-a^{ij}D_{ij}u + b^i D_i u + cu = f \quad \text{in } \Omega, \tag{2.13}$$

$$u = g \quad \text{on } \partial\Omega. \tag{2.14}$$

Theorem 2.5 (Existence and uniqueness, Dirichlet). [21, p. 36, Theorem 7.3]. *Let $\partial\Omega \in C^{2,\alpha}$ $(0 < \alpha < 1)$. Suppose that the coefficients in (2.13) satisfy (2.8) and (2.9), $c \geqslant 0$, $f \in C^\alpha(\overline{\Omega})$, and $g \in C^{2,\alpha}(\overline{\Omega})$. Then the problem (2.13), (2.14) admits a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.*

Remark 2.3. [21, p. 34, Theorem 7.2]. If the condition $\partial\Omega \in C^{2,\alpha}$ is replaced by exterior sphere condition and $g \in C^{2,\alpha}(\overline{\Omega})$ is replaced by the existence of an extension such that $g \in C(\overline{\Omega})$, then the above existence and uniqueness is valid in the space $C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$.

Consider next the linear second order elliptic equation of divergence form:

$$Lu = -D_j(a^{ij}D_i u + d^j u) + b^i D_i u + cu = f + D_i f^i \quad \text{in } \Omega. \tag{2.15}$$

Assume that there exist $\Lambda \geqslant \lambda > 0$ such that

$$\lambda|\xi|^2 \leqslant a^{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n, \tag{2.16}$$

$a^{ij}, d^j, f^i \in C^\alpha(\overline{\Omega})$ $(0 < \alpha < 1)$ and $b^i, c, f \in L^\infty(\Omega)$

$$\frac{1}{\lambda}\left\{ \sum_{i,j} |a^{ij}|_{\alpha;\Omega} + \sum_j |d^j|_{\alpha;\Omega} + \sum_i |b^i|_{L^\infty} + |c|_{L^\infty} \right\} \leqslant \Lambda_\alpha. \tag{2.17}$$

Theorem 2.6 (Interior Schauder $C^{1+\alpha}$ estimates). [67, p. 210, Theorem 8.32]. *Suppose that the coefficients in (2.15) satisfy the assumptions (2.16) and (2.17). Let $u \in C^{1,\alpha}(\Omega)$ $(0 < \alpha < 1)$ be a solution of (2.15). Then for $\Omega' \subset\subset \Omega$, we have*

$$|u|_{1,\alpha;\Omega'} \leqslant C\left(\frac{1}{\lambda}\left(\sum_i |f^i|_{\alpha;\Omega} + |f|_{0;\Omega}\right) + |u|_{0;\Omega}\right), \tag{2.18}$$

where C depends only on $n, \alpha, \Lambda/\lambda, \Lambda_\alpha$ and $\operatorname{dist}\{\Omega', \partial\Omega\}$.

Theorem 2.7 (Global Schauder $C^{1+\alpha}$ estimates, Dirichlet). [67, p. 210, Theorem 8.33]. *Let the assumptions (2.16), (2.17) be in force, and $\partial\Omega \in C^{1,\alpha}$ $(0 < \alpha < 1)$. Suppose that $u \in C^{1,\alpha}(\overline{\Omega})$ is a solution of (2.15) satisfying the boundary condition $u\Big|_{\partial\Omega} = g$ for some $g \in C^{1,\alpha}(\overline{\Omega})$. Then*

$$|u|_{1,\alpha;\Omega} \leqslant C\left(\frac{1}{\lambda}\left(\sum_i |f^i|_{\alpha;\Omega} + |f|_{0;\Omega}\right) + |u|_{0;\Omega} + |g|_{1,\alpha;\Omega}\right), \tag{2.19}$$

where C depends only on $n, \alpha, \Lambda/\lambda, \Lambda_\alpha$ and Ω.

Theorem 2.8 (Global Schauder $C^{1+\alpha}$ estimates, Neumann). *Let the assumptions (2.16), (2.17) be in force, and $\partial\Omega \in C^{1,\alpha}$ $(0 < \alpha < 1)$. Suppose that $u \in C^{1,\alpha}(\overline{\Omega})$ is a solution of (2.15) satisfying the boundary condition $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g$ for some $g \in C^{\alpha}(\overline{\Omega})$. Then*

$$|u|_{1,\alpha;\Omega} \leqslant C\left(\frac{1}{\lambda}\left(\sum_i |f^i|_{\alpha;\Omega} + |f|_{0;\Omega}\right) + |u|_{0;\Omega} + |g|_{\alpha;\Omega}\right), \tag{2.20}$$

where C depends only on $n, \alpha, \Lambda/\lambda, \Lambda_\alpha$ and Ω.

The above theorem is a standard classical result. We outline the proof below:

Since a $C^{1+\alpha}$ diffeomorphism does not change the structure of the equation, one can flatten the boundary. One can then proceed as in the proof of Theorem 2.7 to freeze the leading order coefficients [67, p. 210, (8.84)], so that the problem is essentially reduced to $-\Delta u = f + D_i f^i$ with oblique derivative boundary conditions. After extending f and f_i to the lower half space and subtracting the Newtonian potential (see (8.82) on p. 209 of [67]), the problem is essentially reduce to $-\Delta\psi = 0$ with boundary condition $D\psi(x)\cdot\nu(x) = g(x)$, where $\nu(x)\cdot\mathbf{n} \neq 0$ and $\mathbf{n}$ is the exterior normal, and ν, g are in C^{α}. The $C^{1+\alpha}$ estimates of ψ are covered in [93, Theorem 1].

2.3 A Strong Maximum Principle

Theorem 2.9 (Hopf). ([67, p. 34, Lemma 3.4], see also the remark at the bottom of the page after the proof). *Suppose that Ω satisfies the interior sphere condition at $x = x_0 \in \partial\Omega$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$Lu \equiv -a^{ij}D_{ij}u + b^i D_i u + cu \leqslant 0 \quad \text{in } \Omega, \tag{2.21}$$

where $a^{ij}, b^i, c \in C(\overline{\Omega})$ satisfy the ellipticity condition (2.8) and $c \geqslant 0$. If $u \not\equiv \text{const}$. and takes a non-negative maximum at x_0, then

$$\liminf_{t\to 0} \frac{u(x_0 + t\eta) - u(x_0)}{t} > 0 \tag{2.22}$$

for any direction η such that $\eta \cdot n > 0$, where n is the exterior normal vector.

In the case $c \equiv 0$, the phrase "non-negative maximum" may be replaced by "maximum."

Theorem 2.10 (Strong maximum principle). ([67, p. 35, Theorem 3.5], see also the remark after the proof about locally boundedness of the coefficients). *Suppose that $u \in C^2(\Omega)$ satisfies*

$$Lu \equiv -a^{ij}D_{ij}u + b^i D_i u + cu \leqslant 0 \quad \text{in } \Omega, \tag{2.23}$$

where $a^{ij}, b^i, c \in C(\Omega)$ *satisfy the ellipticity condition* (2.8) *and* $c \geqslant 0$*. If* u *takes a non-negative maximum at an interior point in* Ω*, then* $u \equiv \text{const.}$

In the case $c \equiv 0$*, the phrase "non-negative maximum" may be replaced by "maximum."*

2.4 De Giorgi–Nash–Moser Estimates

In 1957, De Giorgi established Hölder norm estimates for solutions of elliptic equations with measurable coefficients. In 1958, Nash independently obtained similar estimates for parabolic equations. This was a breakthrough for the study of quasilinear elliptic and parabolic equations. In 1960, Moser gave a simplified proof for these estimates, and they are now called the De Giorgi–Nash–Moser estimates.

Consider the following equation of divergence form:

$$-D_j(a^{ij}D_iu) + b^iD_iu + cu = f + D_if^i, \tag{2.24}$$

where we assume that

$$\lambda|\xi|^2 \leqslant a^{ij}\xi_i\xi_j \leqslant \Lambda|\xi|^2, \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n, \tag{2.25}$$

$$\sum_{i,j} \|a^{ij}\|_{L^\infty(\Omega)} + \sum_i \|b^i\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} \leqslant \Lambda. \tag{2.26}$$

Theorem 2.11 (Interior estimates). [21, p. 62, Theorem 2.3]. *Let the assumptions* (2.25), (2.26) *be in force.* $f \in L^{q_*}(\Omega)$, $f^i \in L^q(\Omega)$ *for some* $q > n$*, where* $q_* = nq/(n+q)$*. If* u *is a weak solution of* (2.24)*, then there exist* $C > 0$ *and* $0 < \gamma < 1$ *such that for* $\Omega' \subset\subset \Omega$*,*

$$|u|_{\gamma,\Omega'} \leqslant C\left[|u|_{0,\Omega} + \|f\|_{L^{q_*}(\Omega)} + \sum_i \|f^i\|_{L^q(\Omega)}\right], \tag{2.27}$$

where C *and* γ *depend only on* λ*,* Λ*,* n, q*,* Ω *and* Ω'*.*

Global estimates require the uniform exterior cone condition.

Theorem 2.12 (Global estimates, Dirichlet). [21, p. 65, Theorem 3.5]. *Let the assumptions* (2.25), (2.26) *be in force and the exterior cone condition is satisfied uniformly on* $\partial\Omega$*.* $f \in L^{q_*}(\Omega)$, $f^i \in L^q(\Omega)$ *for some* $q > n$*, where* $q_* = nq/(n+q)$*. If* u *is a weak solution of* (2.24) *with* $u = g\Big|_{\partial\Omega}$ *for some* $g \in C^\alpha(\partial\Omega)$*, then there exist* $C > 0$ *and* $0 < \gamma \leqslant \alpha$ *such that*

$$|u|_{\gamma,\Omega} \leqslant C\left[|u|_{0,\Omega} + \|f\|_{L^{q_*}(\Omega)} + \sum_i \|f^i\|_{L^q(\Omega)} + |g|_{\alpha,\partial\Omega}\right], \tag{2.28}$$

where C *and* γ *depend only on* λ*,* Λ*,* n, q*,* α *and* Ω*.*

The Hölder estimates also extend to conormal boundary conditions.

2.5 L^p Estimates

The $W^{2,p}$ estimate is based on the Calderón–Zygmund decomposition lemma and the singular integral operator theory.

The continuity assumption on the leading order coefficients cannot be dropped.

Consider the operator

$$Lu = -a^{ij}D_{ij}u + b^i D_i u + cu = f \quad \text{in } \Omega. \tag{2.29}$$

Assume that the coefficients in (2.29) satisfy

$$a^{ij}\xi_i\xi_j \geqslant \lambda|\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n, \quad \lambda > 0, \tag{2.30}$$

$$\sum_{ij} \|a^{ij}\|_{L^\infty(\Omega)} + \sum_i \|b^i\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} \leqslant \Lambda, \tag{2.31}$$

$$a^{ij} \in C(\overline{\Omega}) \quad (i, j = 1, 2, \cdots, n). \tag{2.32}$$

Theorem 2.13 (Interior estimates). [21, p. 47, Theorem 4.2]. *Let the assumptions (2.30)–(2.32) be in force. Suppose that $u \in W^{2,p}_{loc}(\Omega)$ $(1 < p < \infty)$ satisfies (2.29) almost everywhere. Then for any $\Omega' \subset\subset \Omega$,*

$$\|u\|_{W^{2,p}(\Omega')} \leqslant C\Big\{\frac{1}{\lambda}\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\Big\},$$

where C depends only on $n, p, \Lambda/\lambda$, $\operatorname{dist}\{\Omega', \partial\Omega\}$ and the modulus of continuity of a^{ij}.

Theorem 2.14 (Global estimates, Dirichlet). [21, p. 48, Theorem 5.4]. *Let the assumptions (2.30)–(2.32) be in force and assume that $\partial\Omega \in C^2$. Suppose that $u \in W^{2,p}(\Omega)$ $(1 < p < \infty)$ satisfies (2.29) almost everywhere and $u = g\Big|_{\partial\Omega}$, where g can be extended to a function on Ω such that $g \in W^{2,p}(\Omega)$. Then*

$$\|u\|_{W^{2,p}(\Omega)} \leqslant C\Big\{\frac{1}{\lambda}\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} + \|g\|_{W^{2,p}(\Omega)}\Big\},$$

where C depends only on $n, p, \Lambda/\lambda$, Ω and the modulus of continuity of a^{ij}.

Remark 2.4. If $\partial\Omega \in C^2$, then the trace operator $W^{2,p}(\Omega) \to W^{2-1/p,p}(\partial\Omega)$ is continuous. So the term $\|g\|_{W^{2,p}(\Omega)}$ may be replaced by $\|g\|_{W^{2-1/p,p}(\partial\Omega)}$.

Remark 2.5. The Schauder estimates imply the existence theorem for a $C^{2,\alpha}$ solution. Similarly, the $W^{2,p}$ estimates imply the existence theorem for a $W^{2,p}$ solution.

Consider next the linear second order elliptic equation of divergence form:

$$Lu = -D_j(a^{ij}D_i u + d^j u) + b^i D_i u + cu = f + D_i f^i \quad \text{in } \Omega. \tag{2.33}$$

Assume that there exist $\Lambda \geqslant \lambda > 0$ such that

$$\lambda|\xi|^2 \leqslant a^{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n, \tag{2.34}$$

$a^{ij} \in C(\overline{\Omega})$, $d^j, b^i, c^i \in L^\infty(\Omega)$ and $f^i \in L^p(\Omega)$, $f \in L^q(\Omega)$ with $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$,

$$\frac{1}{\lambda}\left\{ \sum_j |d^j|_{L^\infty(\Omega)} + \sum_i |b^i|_{L^\infty(\Omega)} + |c|_{L^\infty(\Omega)} \right\} \leqslant \Lambda_\infty. \tag{2.35}$$

Theorem 2.15 (Interior estimates). *Let the assumptions* (2.34) *and* (2.35) *be in force. Suppose that* $u \in W^{1,p}_{loc}(\Omega)$ $(2 < p < \infty)$ *satisfies* (2.33) *in the weak sense. Then for any* $\Omega' \subset\subset \Omega$,

$$\|u\|_{W^{1,p}(\Omega')} \leqslant C\left\{ \frac{1}{\lambda}\left(\|f\|_{L^q(\Omega)} + \sum_i \|f^i\|_{L^p(\Omega)} \right) + \|u\|_{H^1(\Omega)} \right\},$$

where C *depends only on* $n, p, \Lambda/\lambda$, Λ_∞, $\mathrm{dist}\{\Omega', \partial\Omega\}$ *and the modulus of continuity of* a^{ij}.

This theorem is also valid for elliptic systems. In the case $d^j = 0, b^i = 0, c = 0$, $f = 0$, this theorem is a special case of the elliptic system discussed in [21, p. 157, Theorem 2.2], or [59, p. s 71–74].

For nonzero f, we extend f to be zero outside Ω and let $\psi = \Delta^{-1}f$. Using $W^{2,p}$ estimates and embedding, we obtain $\|\nabla\psi\|_{L^p} \leqslant C\|\nabla\psi\|_{W^{1,q}} \leqslant C\|f\|_{L^q}$, so that we can rewrite f as $f = \mathrm{div}(\nabla\psi)$ and combine it with $D_i f^i$.

In the general case we let $\widetilde{f} = f - b^i D_i u - cu$ and $\widetilde{f^i} = f^i + d^i u$. Using $u \in H^1$, we find that $\widetilde{f} \in L^{\min(2,q)}$ and $\widetilde{f^i} \in L^{\min(p,2^*)}$. Thus the theorem is valid for $p_1 = \min(p, 2^*)$. If $p_1 \leqslant n$, a second iteration gives $p_2 = \min(p, p_1^*)$. A finite number of iterations (with shrinking domains) completes the proof.

Theorem 2.16 (Global estimates, Dirichlet). *Let the assumptions* (2.34) *and* (2.35) *be in force and assume that* $\partial\Omega \in C^1$. *Suppose that* $u \in W^{1,p}(\Omega)$ $(2 < p < \infty)$ *satisfies* (2.33) *in the weak sense and* $u = g\Big|_{\partial\Omega}$, *where* g *can be extended to a function on* Ω *such that* $g \in W^{1,p}(\Omega)$. *Then*

$$\|u\|_{W^{1,p}(\Omega)} \leqslant C\left\{ \frac{1}{\lambda}\left(\|f\|_{L^q(\Omega)} + \sum_i \|f^i\|_{L^p(\Omega)} \right) + \|g\|_{W^{1,p}(\Omega)} + \|u\|_{L^2(\Omega)} \right\},$$

where C *depends only on* $n, p, \Lambda/\lambda$, Λ_∞, Ω *and the modulus of continuity of* a^{ij}.

Note that the $\|u\|_{H^1(\Omega)}$ can be estimated in terms of $\|u\|_{L^2}$ under the given structure. The rest of the proof is similar to that of the global $W^{2,p}$ estimates.

2.6 Krylov–Safonov Estimates

Krylov and Safonov obtained the Hölder estimate for equations in non-divergence form in 1980. The proof relies on the Alexandroff–Bakelman–Pucci maximum principle.

Now we discuss the following elliptic equation:

$$Lu \equiv -a^{ij}D_{ij}u + b^i u + cu = f, \tag{2.36}$$

where we assume that

$$(a^{ij}) \geqslant 0 \quad \text{in } \Omega, \tag{2.37}$$

$$\|c/\mathcal{D}^*\|_{L^n(\Omega)} + \sum_i \|b^i/\mathcal{D}^*\|_{L^n(\Omega)} \leqslant B, \tag{2.38}$$

where $\mathcal{D}^* = [\det(a^{ij})]^{1/n}$.

Theorem 2.17 (Alexandroff–Bakelman–Pucci maximum principle). [21, p. 86, Theorem 1.9]. *Suppose that $u \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ satisfies the inequality $Lu \leqslant f$ almost everywhere in Ω and the coefficients satisfy (2.37), (2.38) and*

$$c \geqslant 0 \quad \text{on } \Omega. \tag{2.39}$$

Then

$$\sup_{\Omega} u(x) \leqslant \sup_{\partial\Omega} u(x) + C\left\|\frac{f^+}{\mathcal{D}^*}\right\|_{L^n(\Omega)}, \tag{2.40}$$

where C depends only on n, B and $\operatorname{diam}\Omega$.

Theorem 2.18 (Interior C^α estimates). [21, p. 95, Theorem 2.5]. *Let the coefficients of L satisfy the uniform ellipticity condition with elliptic constants $\Lambda > \lambda > 0$. Suppose that $u \in W^{2,n}_{loc}(\Omega)$ satisfies $Lu = f$ almost everywhere in Ω and $f/\lambda \in L^n(\Omega)$. Then there exist $C > 0$ and $0 < \alpha < 1$ such that, for any $B_{R_0}(y) \subset \Omega$ and $0 < R \leqslant R_0$,*

$$\operatorname*{osc}_{B_R(y)} u \leqslant C\left(\frac{R}{R_0}\right)^{\alpha}\Big[|u|_{L^\infty(\Omega)} + R_0\|f/\lambda\|_{L^n(\Omega)}\Big],$$

where C and α depend only on n and B.

Theorem 2.19 (Global C^α estimates, Dirichlet). [21, p. 97, Theorem 3.2]. *Let the coefficients of L satisfy the uniform ellipticity condition with constants $\Lambda > \lambda > 0$ and Ω satisfy the uniform exterior sphere condition. Suppose that $u \in W^{2,n}_{loc}(\Omega)$*

satisfies $Lu = f$ *almost everywhere in* Ω, $u = g|_{\partial\Omega}$, *where* $g \in C^{\beta}(\partial\Omega)$ *and* $f/\lambda \in L^n(\Omega)$. *Then there exist* $C > 0$ *and* $0 < \alpha \leqslant \beta$ *such that,*

$$|u|_{\alpha,\Omega} \leqslant C\Big[|u|_{0,\Omega} + |g|_{\beta,\Omega} + \|f/\lambda\|_{L^n(\Omega)}\Big],$$

where C *and* α *depend only on* n, β *and* B.

2.7 Poincaré's Inequality and Embedding Theorems

Theorem 2.20. (Embedding theorem [1, Theorem 5.4]). *Let* Ω *be a bounded domain,* k *a positive integer, and* $1 \leqslant p \leqslant +\infty$.

(a) If Ω *satisfies an interior cone condition, then*

(i) for $kp < n$, $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \in [p, p^*]$, *where* $p^* = np/(n - kp)$;
(ii) for $kp = n$, $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \in [p, +\infty)$.

(b) If Ω *is locally Lipschitz, then for* $kp > n$,

$$W^{k,p}(\Omega) \hookrightarrow C^{k-1-[n/p],\alpha}(\overline{\Omega}), \qquad \forall \alpha \in [0, \alpha_0],$$

where $\alpha_0 = [n/p] + 1 - (n/p)$ *if* n/p *is not an integer, and* α_0 *can be chosen to be any positive number less than 1 if* n/p *is an integer.*

In all the above cases, the embedding constants depend on n, k, p, Ω.

Theorem 2.21. (Compact embedding theorem [1, Theorem 6.2]). *Let* Ω *be a bounded domain,* k *a positive integer, and* $1 \leqslant p \leqslant +\infty$.

(a) If Ω *satisfies an interior cone condition, then the following embeddings are compact:*

(i) for $kp < n$, $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \in [p, p^*)$, *where* $p^* = np/(n - kp)$;
(ii) for $kp = n$, $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \in [p, +\infty)$.

(b) If Ω *is locally Lipschitz, then for* $kp > n$, *the following embedding is compact:*

$$W^{k,p}(\Omega) \hookrightarrow C^{k-1-[n/p],\alpha}(\overline{\Omega}), \qquad \forall \alpha \in [0, \alpha_0),$$

where α_0 *is the same as in the above theorem.*

Poincaré's inequality:

Theorem 2.22. (Variational Principle for the principle eigenvalue, [30, p. 336, Theorem 2]). *Let* Ω *be a bounded domain. If* $u \in H_0^1(\Omega)$, *then*

$$\lambda_1 \int_{\Omega} u^2 dx \leqslant \int_{\Omega} |\nabla u|^2 dx,$$

where $\lambda_1 > 0$ *is the first eigenvalue of the problem:*

$$-\Delta\phi = \lambda\phi, \quad \text{in } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

Theorem 2.23. [21, p. 213, Theorem 3.1]. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$.

(1) If $u \in W_0^{1,p}(\Omega)$, $1 \leqslant p < +\infty$, *then*

$$\int_\Omega |u|^p dx \leqslant C(n,p,\Omega) \int_\Omega |\nabla u|^p dx.$$

(2) If Ω *is a bounded domain with* $\partial\Omega$ *Lipschitz continuous, then for* $u \in W^{1,p}(\Omega)$, $1 \leqslant p < +\infty$,

$$\int_\Omega |u - u_\Omega|^p dx \leqslant C(n,p,\Omega) \int_\Omega |\nabla u|^p dx, \qquad \Big(u_\Omega := \frac{1}{|\Omega|}\int_\Omega u dx\Big).$$

2.8 Exercises

2.1. Let Ω be a bounded domain in $\mathbb{R}^n$. For any $0 \leqslant \beta < \alpha \leqslant 1$, show that the embedding $C^\alpha(\overline{\Omega}) \hookrightarrow C^\beta(\overline{\Omega})$ is compact.

2.2. Let Ω be a bounded domain with $\partial\Omega \in C^{2+\alpha}$. Consider the linear second order elliptic equation of non-divergence form:

$$\begin{aligned} -a_k^{ij} D_{ij} u_k + b_k^i D_i u_k + c_k u_k &= f_k \quad \text{in } \Omega, \\ u_k &= g_k \quad \text{on } \partial\Omega, \end{aligned}$$

where g_k is assumed to have been extended to $\overline{\Omega}$, as usual.

(a) Assume that there exist $\Lambda \geqslant \lambda > 0$, independent of k, such that

$$\lambda|\xi|^2 \leqslant a_k^{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n,$$

$a_k^{ij}, b_k^i, c_k \in C^\alpha(\overline{\Omega})$ $(0 < \alpha < 1)$ and

$$\frac{1}{\lambda}\Big\{\sum_{i,j} |a_k^{ij}|_{\alpha;\Omega} + \sum_i |b_k^i|_{\alpha;\Omega} + |c_k|_{\alpha;\Omega} + |f_k|_{\alpha;\Omega}\Big\} + |g_k|_{2+\alpha;\Omega} \leqslant \Lambda;$$

(b) Assume that a_k^{ij}, b_k^i, c_k, f_k, and g_k converge weakly in $L^1(\Omega)$ to a^{ij}, b^i, c, f, and g, respectively;

(c) Assume that

$$\sup_{1\leqslant k<\infty} \|u_k\|_{L^\infty(\Omega)} < \infty;$$

(1) Prove that there is a subsequence $\{u_{k_l}\}$ that converges in $C^{2+\beta}(\overline{\Omega})$ to a solution of the limiting problem, and that the limit belongs to the space $C^{2+\alpha}(\overline{\Omega})$.
(2) If we further assume that $c_k \geqslant 0$ for all k, show that the assumption (c) is no longer necessary. Furthermore, show that the convergence is valid for the sequence $\{u_k\}$, not just limited to a subsequence.

2.3. Exercise 2.2 involves the Schauder estimate. Form and prove your result using $W^{2,p}$ estimate.

Chapter 3
A Review of Parabolic Theories

All theorems in Chap. 2 have their parabolic version. As mentioned at the beginning of Chap. 2, the books [21] and [20] are not intimidating even for beginners and therefore are excellent textbooks for beginning graduate students. Elliptic theories are introduced in [21] and the parabolic theories are introduced in [20]. The results from this chapter can also be found in [94] and [29].

3.1 Weak Solutions and a Weak Maximum Principle

Let Ω be a domain in $\mathbb{R}^n$. Consider parabolic equations of divergence form in $\Omega_T \equiv \Omega \times (0, T]$:

$$Lu \equiv u_t - D_j(a^{ij} D_i u + d^j u) + (b^i D_i u + cu) = f + \sum_i D_i f^i. \tag{3.1}$$

As before, we assume that there exist positive constants λ, Λ such that

$$\lambda |\xi|^2 \leqslant a^{ij}(x,t)\xi_i \xi_j \leqslant \Lambda |\xi|^2, \qquad \forall \xi \in \mathbb{R}^n, (x,t) \in \Omega_T. \tag{3.2}$$

Definition 3.1. For $f, c, f^i, d^j, b^i \in L^2(\Omega_T)$ and $g \in L^2[0, T; H^1(\Omega)]$, $u_0 \in H^1(\Omega)$, we say that $u \in L^2[0, T; H^1(\Omega)] \cap L^\infty[0, T; L^2(\Omega)]$ (this space is also known as $V_2(\Omega_T)$) is a *weak solution* of the Dirichlet problem

$$\begin{cases} Lu = f + \sum_i D_i f^i & \text{in } \Omega_T, \\ u = g & \text{on } \Gamma_T \equiv \partial\Omega \times [0, T], \\ u\Big|_{t=0} = u_0(x), \end{cases} \tag{3.3}$$

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if u satisfies

$$\begin{cases} \int_0^T\int_\Omega \Big\{ -uv_t + (a^{ij}D_iu + d^ju)D_jv + (b^iD_iu + cu)v\Big\} dx\,dt - \int_\Omega u_0v\Big|_{t=0} dx \\ = \int_0^T\int_\Omega fvdxdt - \int_0^T\int_\Omega f^iD_ivdxdt, \ \forall v \in C^1(\overline{\Omega}_T), \ v=0 \text{ on } \Gamma_T \cup \{t=T\}, \\ u-g \in L^2[0,T;H_0^1(\Omega)]. \end{cases} \tag{3.4}$$

Definition 3.2. We define *weak subsolution* and *weak supersolution* by replacing the equality above with "$\leqslant$" and "$\geqslant$" respectively, and further requiring the test function v to be non-negative.

Theorem 3.1 (Weak maximum principle). *Let the assumption* (3.2) *be in force, and* $f, f^i \in L^2(\Omega_T)$, $c, b^i, d^i \in L^\infty(\Omega_T)$ *and* $g \in L^2[0,T;H^1(\Omega)]$, $u_0 \in H^1(\Omega)$, *and*

$$\int_0^T\int_\Omega (c\phi + d^iD_i\phi)dxdt \geqslant 0, \ \ \forall \phi \in C^1(\overline{\Omega}_T), \phi = 0 \text{ on } \Gamma_T, \ \phi \geqslant 0,$$
$$\int_0^T\int_\Omega (f\phi - f^iD_i\phi)dxdt \leqslant 0, \ \ \forall \phi \in C^1(\overline{\Omega}_T), \phi = 0 \text{ on } \Gamma_T, \ \phi \geqslant 0,$$
$$g \leqslant 0 \quad \text{on } \Gamma_T, \quad u_0 \leqslant 0 \quad \text{on } \Omega.$$

If $u \in L^2[0,T;H^1(\Omega)] \cap L^\infty[0,T;L^2(\Omega)]$ *is a subsolution, then*

$$u \leqslant 0 \quad \text{in } \Omega_T.$$

This theorem can be derived directly from [94, p. 128, Theorem 6.25].

Remark 3.1. The existence of a weak solution requires higher regularity on the coefficients.

3.2 Schauder Theory

The Schauder theory is also a powerful tool for the classical solution for parabolic equations.

Let Ω be a bounded domain. Consider the linear second order elliptic equation of non-divergence form:

$$Lu = u_t - a^{ij}D_{ij}u + b^iD_iu + cu = f \quad \text{in } \Omega_T \equiv \Omega \times [0,T]. \tag{3.5}$$

Assume that there exist $\Lambda \geqslant \lambda > 0$ such that

$$\lambda|\xi|^2 \leqslant a^{ij}(x,t)\xi_i\xi_j \leqslant \Lambda|\xi|^2, \quad \forall (x,t) \in \Omega_T, \xi \in \mathbb{R}^n, \tag{3.6}$$

$a^{ij}, b^i, c \in C^{\alpha,\alpha/2}(\overline{\Omega}_T)$ $(0 < \alpha < 1)$ and

$$\frac{1}{\lambda}\left\{ \sum_{i,j} |a^{ij}|_{C^{\alpha,\alpha/2}(\overline{\Omega}_T)} + \sum_i |b^i|_{C^{\alpha,\alpha/2}(\overline{\Omega}_T)} + |c|_{C^{\alpha,\alpha/2}(\overline{\Omega}_T)} \right\} \leqslant \Lambda_\alpha. \tag{3.7}$$

Theorem 3.2 (Interior Schauder estimates). [94, p. 59, Theorem 4.9]. *Suppose that the coefficients in* (3.5) *satisfy the assumptions* (3.6) *and* (3.7). *Let* $u \in C^{\alpha,\alpha/2}(\overline{\Omega}_T)$ $(0 < \alpha < 1)$ *be a solution of* (3.5). *Then for* $\Omega' \subset\subset \Omega$ *and* $\eta > 0$, *we have*

$$|u|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}'\times[\eta,T])} \leqslant C\left(\frac{1}{\lambda}|f|_{C^{\alpha,\alpha/2}(\overline{\Omega}_T)} + |u|_{C(\overline{\Omega}_T)}\right), \tag{3.8}$$

where C *depends only on* $n, \alpha, \Lambda/\lambda, \Lambda_\alpha$, $\mathrm{dist}\{\Omega', \partial\Omega\}$, T *and* η.

Theorem 3.3 (Global Schauder estimates, Dirichlet). [94, p. 78, Theorem 4.28]. *Let the assumptions* (3.6), (3.7) *be in force, and* $\partial\Omega \in C^{2,\alpha}$ $(0 < \alpha < 1)$. *Let* $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_T)$ *be a solution of* (3.5) *satisfying the boundary condition* $u\Big|_{\Gamma_T} = g$ *for some* $g \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_T)$ *and the initial condition* $u\Big|_{t=0} = u_0(x)$ *for some* $u_0 \in C^{2,\alpha}(\overline{\Omega})$. *Furthermore, assume that* u *satisfies the second order compatibility conditions:*

$$u_0(x) = g(x,0),\ x \in \partial\Omega,$$
$$g_t - a^{ij}D_{ij}u + b^iD_iu + cu - f = 0,\ x \in \partial\Omega,\ t = 0.$$

Then

$$|u|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_T)} \leqslant C\left(\frac{1}{\lambda}|f|_{C^{\alpha,\alpha/2}(\overline{\Omega}_T)} + |g|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_T)} + |u_0|_{C^{2+\alpha}(\overline{\Omega})}\right), \tag{3.9}$$

where C *depends only on* $n, \alpha, \Lambda/\lambda, \Lambda_\alpha$ *and* Ω_T.

Theorem 3.4 (Global Schauder estimates, Neumann). [94, p. 79, Theorem 4.31]. *Let the assumptions* (3.6), (3.7) *be in force, and* $\partial\Omega \in C^{2,\alpha}$ $(0 < \alpha < 1)$. *Let* $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_T)$ *be a solution of* (3.5) *satisfying the boundary condition* $\frac{\partial u}{\partial n}\Big|_{\Gamma_T} = g$ *for some* $g \in C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_T)$ *and the initial condition* $u\Big|_{t=0} = u_0(x)$ *for some* $u_0 \in C^{2,\alpha}(\overline{\Omega})$. *We further assume the first order compatibility condition:*

$$\frac{\partial}{\partial n}u_0(x) = g(x,0),\ x \in \partial\Omega.$$

Then

$$|u|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_T)} \leqslant C\left(\frac{1}{\lambda}|f|_{C^{\alpha,\alpha/2}(\overline{\Omega}_T)} + |g|_{C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_T)} + |u_0|_{C^{2+\alpha}(\overline{\Omega})}\right), \tag{3.10}$$

where C depends only on $n, \alpha, \Lambda/\lambda, \Lambda_\alpha$ and Ω.

Remark 3.2. These estimates also extend to other oblique boundary conditions.

Remark 3.3. The combined boundary-interior estimates (without initial condition) are also valid.

Now consider the Dirichlet problem

$$u_t - a^{ij}D_{ij}u + b^iD_iu + cu = f \qquad \text{in } \Omega_T, \tag{3.11}$$

$$u = g \qquad \text{on } \Gamma_T, \tag{3.12}$$

$$u\Big|_{t=0} = u_0(x) \qquad \text{for } x \in \Omega. \tag{3.13}$$

Theorem 3.5 (Existence and uniqueness, Dirichlet). [94, p. 94, Theorem 5.14]. *Let $\partial\Omega \in C^{2+\alpha}$ $(0 < \alpha < 1)$. Suppose that the coefficients in (3.11) satisfy (3.6) and (3.7), $f \in C^{\alpha,\alpha/2}(\overline{\Omega}_T)$, $g \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_T)$, and $u_0 \in C^{2+\alpha}(\overline{\Omega})$ satisfies the second order compatibility conditions. Then the problem (3.11)–(3.13) admits a unique solution $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_T)$.*

3.3 A Strong Maximum Principle

Theorem 3.6. [94, p. 10, Lemma 2.6]. *Suppose that Ω satisfies the interior sphere condition at $x = x_0 \in \partial\Omega$ and $u \in C^2(\Omega_T) \cap C(\overline{\Omega}_T)$ satisfies*

$$Lu \equiv u_t - a^{ij}D_{ij}u + b^iD_iu + cu \leqslant 0 \quad \text{in } \Omega_T, \tag{3.14}$$

where $a^{ij}, b^i, c \in C(\overline{\Omega}_T)$ satisfy the ellipticity condition (2.8) and $c \geqslant 0$. If $u \not\equiv \text{const}$. and takes a non-negative maximum at (x_0, t_0), then

$$\liminf_{\sigma\to 0} \frac{u(x_0 + \sigma\eta, t_0) - u(x_0, t_0)}{\sigma} > 0 \tag{3.15}$$

for any direction η such that $\eta \cdot n > 0$, where n is the exterior normal vector.

In the case $c \equiv 0$, the phrase "non-negative maximum" may be replaced by "maximum."

Remark 3.4. Ω_T may be replaced by a general set $Q \subset \mathbb{R}^n \times [0, T]$ which satisfies the interior ellipsoid condition (see [94]).

Theorem 3.7 (Strong maximum principle). [94, p. 13, Theorem 2.9]. *Suppose that $u \in C^2(\Omega_T)$ satisfies*

$$Lu \equiv u_t - a^{ij} D_{ij} u + b^i D_i u + cu \leqslant 0 \quad \textit{in } \Omega_T, \tag{3.16}$$

where $a^{ij}, b^i, c \in C(\Omega_T)$ satisfy the ellipticity condition (3.6) and $c \geqslant 0$. If u takes a non-negative maximum at a parabolic interior point in Ω_T, then $u \equiv \text{const}$.

In the case $c \equiv 0$, the phrase "non-negative maximum" may be replaced by "maximum."

3.4 De Giorgi–Nash–Moser Estimates

Consider the following equation of divergence form:

$$u_t - D_j(a^{ij} D_i u) + b^i D_i u + cu = f + D_i f^i, \tag{3.17}$$

where we assume that, for some $q > n + 2$

$$\lambda |\xi|^2 \leqslant a^{ij} \xi_i \xi_j \leqslant \Lambda |\xi|^2, \quad \forall (x,t) \in \Omega_T,\ \xi \in \mathbb{R}^n, \tag{3.18}$$

$$\sum_i \|b^i\|_{L^q(\Omega_T)} + \|c\|_{L^{q/2}(\Omega_T)} \leqslant \Lambda, \quad c(x,t) \geqslant -\Lambda. \tag{3.19}$$

Theorem 3.8 (Interior estimates). ([20, p. 142, Theorem 4.3], or, stated in a more general form with different integration weight in x and t, [83, p. 204, Theorem 10.1]). *Let the assumptions (3.18), (3.19) be in force. $f \in L^{q_*}(\Omega_T)$, $f^i \in L^q(\Omega_T)$ for some $q > n + 2$, where $q_* = (n+2)q/(n+2+q)$. If u is a weak solution of (3.17), then there exist $C > 0$ and $0 < \gamma < 1$ such that for $\Omega'_T \equiv \Omega' \times [\eta, T] \subset \Omega_T$,*

$$|u|_{C^{\gamma,\gamma/2}(\overline{\Omega}'_T)} \leqslant C \left[|u|_{C(\overline{\Omega}_T)} + \|f\|_{L^{q_*}(\Omega_T)} + \sum_i \|f^i\|_{L^q(\Omega_T)} \right], \tag{3.20}$$

where C and γ depend only on λ, Λ, n, q, Ω_T and Ω'_T.

Global estimates require the uniform exterior cone condition.

Theorem 3.9 (Global estimates, Dirichlet). ([20, p. 145, Theorem 6.2], or, with different integration weight in x and t, [83, p. 204, Theorem 10.1]). *Let the assumptions (3.18), (3.19) be in force and the exterior cone condition be satisfied uniformly on $\partial\Omega$. Let $f \in L^{q_*}(\Omega_T)$, $f^i \in L^q(\Omega_T)$ for some $q > n+2$. If u is a weak solution of (3.17) with $u = g\Big|_{\Gamma_T}$ for some $g \in C^{\alpha,\alpha/2}(\partial\Omega)$ and $u\Big|_{t=0} = u_0(x)$ for some $u_0 \in C^\alpha(\overline{\Omega})$ satisfying the zeroth order compatibility condition, then there exist $C > 0$ and $0 < \gamma \leqslant \alpha$ such that*

$$|u|_{C^{\gamma,\gamma/2}(\overline{\Omega}_T)} \leqslant C\left[|g|_{C^{\alpha,\alpha/2}(\partial\Omega)} + |u_0|_{C^{\alpha}(\overline{\Omega})} + \|f\|_{L^{q*}(\Omega_T)} + \sum_i \|f^i\|_{L^q(\Omega_T)}\right], \tag{3.21}$$

where C and γ depend only on λ, Λ, n, q, α and Ω_T.

Remark 3.5. The global estimate is also valid for conormal boundary conditions [94, p. 139, Theorem 6.44].

3.5 L^p Estimates

The $W^{2,p}$ estimate is based on the Calderón–Zygmund decomposition lemma and the singular integral operator theory.

Similar to the elliptic equations, the continuity assumption on the leading order coefficients cannot be dropped.

Consider the operator

$$Lu = u_t - a^{ij}D_{ij}u + b^i D_i u + cu = f \quad \text{in } \Omega_T. \tag{3.22}$$

Assume that the coefficients in (3.22) satisfy

$$a^{ij}\xi_i\xi_j \geqslant \lambda|\xi|^2, \quad \forall(x,t) \in \Omega_T,\ \xi \in \mathbb{R}^n, \quad \lambda > 0, \tag{3.23}$$

$$\sum_{ij} \|a^{ij}\|_{L^\infty(\Omega_T)} + \sum_i \|b^i\|_{L^\infty(\Omega_T)} + \|c\|_{L^\infty(\Omega_T)} \leqslant \Lambda, \tag{3.24}$$

$$a^{ij} \in C(\overline{\Omega}_T) \quad (i,j = 1,2,\cdots,n). \tag{3.25}$$

Theorem 3.10 (Interior estimates). [94, p. 172, Theorem 7.13], or [20, p. 111, Theorem 3.2]. *Let the assumptions (3.23)–(3.25) be in force. Suppose that $u \in W^{2,1,p}_{loc}(\Omega_T)$ $(1 < p < \infty)$ satisfies (3.22) almost everywhere. Then for any $\Omega'_T \equiv \Omega' \times (\eta, T] \subset\subset \Omega_T$,*

$$\|u\|_{W^{2,1,p}(\Omega'_T)} \leqslant C\Big\{\frac{1}{\lambda}\|f\|_{L^p(\Omega_T)} + \|u\|_{L^p(\Omega_T)}\Big\},$$

where C depends only on $n, p, \Lambda/\lambda$, Ω'_T, Ω_T and the modulus of continuity of a^{ij}.

Theorem 3.11 (Global estimates, Dirichlet). [94, p. 176, Theorem 7.17], or [20, p. 113, Theorem 4.2]. *Let the assumptions (3.23)–(3.25) be in force and assume that $\partial\Omega \in C^2$. Suppose that $u \in W^{2,1,p}(\Omega)$ $(1 < p < \infty)$ satisfies (3.22) almost everywhere and $u = g\Big|_{\Gamma_T}$, where g can be extended to a function on Ω_T such that $g \in W^{2,1,p}(\Omega_T)$. Assume further that $u\Big|_{t=0} = u_0(x)$ where $u_0 \in W^{2,p}(\Omega)$ satisfying the zeroth order compatibility condition. Then*

$$\|u\|_{W^{2,1,p}(\Omega_T)} \leqslant C\Big\{\frac{1}{\lambda}\Big(\|f\|_{L^p(\Omega_T)} + \|g\|_{W^{2,1,p}(\Omega_T)}\Big) + \|u_0\|_{W^{2,p}(\Omega)}\Big\},$$

where C depends only on $n, p, \Lambda/\lambda$, Ω_T and the modulus of continuity of a^{ij}.

3.6 Krylov–Safonov Estimates

Krylov and Safonov obtained the Hölder estimate for equations in non-divergence form in 1980, which also applies to parabolic equations. It is also based on the parabolic version of the Alexandroff–Bakelman–Pucci maximum principle

Consider

$$Lu \equiv u_t - a^{ij}D_{ij}u + b^i u + cu = f, \tag{3.26}$$

where we assume that

$$\lambda I \leqslant (a^{ij}) \leqslant \Lambda I \quad \text{in } \Omega_T, \tag{3.27}$$

$$\|c\|_{L^\infty(\Omega_T)} + \sum_i \|b^i\|_{L^\infty(\Omega_T)} \leqslant B. \tag{3.28}$$

Theorem 3.12 (Interior C^α estimates). [94, p. 186, Corollary 7.26], or [20, p. 170, Theorem 3.4]. *Let the coefficients of L satisfy (3.27) and (3.28). Suppose that $u \in W^{2,1,n+1}_{loc}(\Omega_T)$ satisfies $Lu = f$ almost everywhere in Ω and $f/\lambda \in L^{n+2}(\Omega_T)$. Then there exist $C > 0$ and $0 < \alpha < 1$ such that, for any $Q_{R_0}(y,\tau) \subset \Omega_T$ and $0 < R \leqslant R_0$,*

$$\operatorname*{osc}_{Q_R(y,\tau)} u \leqslant C\left(\frac{R}{R_0}\right)^\alpha \Big[|u|_{L^\infty(\Omega_T)} + R_0\|f/\lambda\|_{L^{n+1}(\Omega_T)}\Big],$$

where C and α depend only on n and B.

Theorem 3.13 (Global C^α estimates, Dirichlet). [94, p. 187, Corollary 7.30], or [20, p. 171, Theorem 4.1]. *In addition to the assumptions in the above theorem, we assume $\partial\Omega$ satisfies a uniform exterior cone condition and that u is Hölder continuous on the lateral boundary and on the initial manifold and zeroth order compatibility condition on $\partial\Omega \times \{t = 0\}$, then the above estimates extend to $\overline{\Omega}_T$.*

The Hölder estimates for the case of oblique boundary condition can be found in [94, p. 192, Corollary 7.36].

3.7 Embedding Theorems

The parabolic version of the embedding theorems is very similar to the elliptic version when the t derivatives is considered "half the order" of x-derivatives. There are also embedding theorems for different weights in x and t directions. We only list one here.

In this theorem, we only need to remember that t-direction "takes two-dimensions" when compared with the elliptic version.

Theorem 3.14. *Let $u \in W^{2,1,p}(\Omega_T)$, $\partial\Omega \in C^2$. Then*

(1) [83, p. 80, Lemma 3.3 with $r = 0, s = 1, l = 1$ in (3.15)], or [20, p. 29, Theorem 2.3]. If $1 \leqslant p < n+2$, then for $q = \dfrac{(n+2)p}{n+2-p}$,

$$\|\nabla_x u\|_{L^q(\Omega_T)} \leqslant C\|u\|_{W^{2,1,p}(\Omega_T)};$$

(2) [83, p. 80, Lemma 3.3 with $r = 0, s = 0, l = 1$ in (3.15)], or [20, p. 29, Theorem 2.3]. If $1 \leqslant p < (n+2)/2$, then for $q_1 = \dfrac{(n+2)p}{n+2-2p}$,

$$\|u\|_{L^{q_1}(\Omega_T)} \leqslant C\|u\|_{W^{2,1,p}(\Omega_T)};$$

(3) [83, p. 80, Lemma 3.3 with $r = 0, s = 0$ and $s = 1, l = 1$ in (3.16)], or [20, p. 38, Theorem 3.4]. If $p > n+2$, then for $\alpha = 1 - \dfrac{n+2}{p}$,

$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_T)} \leqslant C\|u\|_{W^{2,1,p}(\Omega_T)},$$

where C depends on n, p, Ω and lower bounds of T.

The next theorem deals with embeddings with different weights for t and x.

Theorem 3.15. [94, p. 110, Theorem 6.9], or [20, p. 43, Theorem 4.1]. *Let $u \in L^2([0,T]; H_0^1(\Omega)) \cap L^\infty([0,T]; L^2(\Omega))$. Then $u \in L^{2(n+2)/n}(\Omega_T)$, and*

$$\|u\|_{L^{2(n+2)/n}(\Omega_T)} \leqslant C \sup_{0<t<T} \|u(\cdot,t)\|_{L^2(\Omega)}^{2/(n+2)} \|\nabla_x u\|_{L^2(\Omega_T)}^{n/(n+2)},$$

where C depends only n.

Since one x derivative is approximately "half t derivative," it may be necessary in some situations to use fractional derivatives in the study of parabolic equations. The fractional α $(0 < \alpha < 1)$ derivative in t direction as follows: A function u is said to have its fractional α derivative in t in L^p space if and only if the following

$$\Big\{\int_\Omega \int_0^T \int_0^T \Big(\frac{|u(x,t) - u(x,\tau)|}{|t-\tau|^\alpha}\Big)^p \frac{dt d\tau}{|t-\tau|} dx\Big\}^{1/p}$$

is finite. If one allows fractional derivatives, then the embedding theorems can also be extended to these spaces.

3.8 Exercises

3.1. Let Ω be a bounded domain and the assumptions of Theorem 3.3 be in force. Furthermore, assume that the coefficients a^{ij}, b^i, c, the right-hand side f, and the boundary data g are all independent of t. Assume further that $c \geqslant c_0 > 0$. Prove that, for any $0 < \beta < \alpha$, the solution to the parabolic equation converges in $C^{2+\beta}(\overline{\Omega})$ to the solution of the corresponding elliptic problem as $t \to \infty$.

3.2. Now we replace the assumption $c \geqslant c_0 > 0$ by

$$\int_0^\infty |u_t(x,t)|^2 dxdt < \infty.$$

Prove that the corresponding elliptic problem admits a solution in $C^{2+\alpha}(\overline{\Omega})$ and that a subsequence $u(\cdot, t_j)$ converges to this elliptic solution as $t_j \to \infty$ in $C^{2+\beta}(\overline{\Omega})$, for any $0 < \beta < \alpha$. This limit solution is called *the ω-limit*.

3.3. If in the above problem the solution to the elliptic problem is unique, prove that $u(\cdot, t)$ converges in $C^{2+\beta}(\overline{\Omega})$ to this elliptic solution as $t \to \infty$, for any $0 < \beta < \alpha$.

3.4. Let Ω be a bounded domain and the assumptions of Theorem 3.3 be in force (except the assumptions on the initial value). Furthermore, assume that the coefficients a^{ij}, b^i, c, the right-hand side f, and the boundary data g are all period functions in t with period T. Assume further that $c \geqslant c_0 > 0$. Prove that there exists a unique periodic solution $u \in C^{2+\alpha, 1+\alpha/2}(\overline{\Omega} \times (-\infty, \infty))$ to the parabolic equation with period T.

Chapter 4
A Review of Fixed Point Theorems

We collect in this chapter several fixed point theorems, which are useful for proving existence of solutions to nonlinear equations and systems.

4.1 Fixed Point Theorems

Theorem 4.1 (Contraction mapping principle). ([67, p. 74, Theorem 5.1], see also the note after the Theorem for replacing the whole space by a closed subset). *Let X be a Banach space and let K be a closed convex set in X. If M is a mapping defined on K such that*

$$Mx \in K \quad \forall x \in K, \tag{4.1}$$

$$\sup_{x,y\in K,\, x\neq y} \frac{\|Mx - My\|}{\|x-y\|} < 1, \tag{4.2}$$

then M has a unique fixed point in K.

Contraction mapping principle is powerful for solving evolution problems. See Exercise 4.1.

Theorem 4.2 (Schauder fixed point theorem). [67, p. 280, Corollary 11.2]. *Let X be a Banach space and let K be a* bounded *closed convex set in X. If M is a mapping on K such that*

$$Mx \in K \quad \forall x \in K, \tag{4.3}$$

$$M \text{ is continuous}, \tag{4.4}$$

$$\overline{MK} \text{ is compact}. \tag{4.5}$$

Then M has at least one fixed point in K.

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Theorem 4.3 (Leray–Schauder fixed point theorem). [67, p. 280, Theorem 11.3]. *Let X be a Banach space and M a mapping on X such that*

$$M : X \to X \text{ is continuous,} \tag{4.6}$$

$$M \text{ is compact, i.e., for any bounded set } B, \overline{MB} \text{ is a compact set,} \tag{4.7}$$

$$\text{the set } \{x \mid x = \lambda M x \text{ for some } \lambda \in [0,1]\} \text{ is bounded in } X. \tag{4.8}$$

Then M has at least one fixed point in X.

4.2 Exercises

If contraction mapping principle is used, then existence and uniqueness are obtained at the same time. For evolution equations, it is often possible to use this argument because for small time, the solution is expected to be close to the initial value. Once this done, the solution can usually be extended further until it can no longer be extended, or up to blow-up time.

4.1. Consider the systems

$$\begin{aligned} u_t - \Delta u &= f(u,x) && \text{in } \Omega \times (0,T), \\ u &= 0 && \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) && \text{in } \Omega, \end{aligned}$$

and

$$\begin{aligned} u_t - \Delta u &= f(u,x) && \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} &= g(u,x) && \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) && \text{in } \Omega. \end{aligned}$$

Assume that f is Lipschitz continuous on $\mathbb{R} \times \overline{\Omega}$, g is Lipschitz continuous on $\mathbb{R} \times \partial\Omega$, and $u_0 \in C^{\alpha}(\overline{\Omega})$, where Ω is a bounded set with $\partial\Omega \in C^{2+\alpha}$. Assume that the zeroth order compatibility condition is satisfied for the first system and the first order compatibility condition is satisfied for the second system. Prove that

1. Each of these systems admits a unique classical solution, locally in time (i.e., for a small time interval).
2. Suppose that the solution exists for $0 < t < S$, and $|u|_{L^\infty(\Omega \times (0,S))} < \infty$. Prove that $u \in C^{2+\alpha, 1+\alpha/2}(\overline{\Omega} \times [\eta, S])$ for any $\eta > 0$.
3. Establish a comparison principle for each of these systems: If u_1 and u_2 are classical sub- and super-solutions, respectively, and $u_1 \leqslant u_2$ on the parabolic boundary, prove that $u_1 \leqslant u_2$ in $\overline{\Omega} \times [0,T]$.

4.2. Consider the system

$$
\begin{aligned}
u_t - \Delta u &= \int_\Omega b(y,t)u(y,t)dy && \text{in } \Omega \times (0,T), \\
u &= 0 && \text{on } \partial\Omega \times (0,T), \\
u(x,0) &= u_0(x) && \text{in } \Omega,
\end{aligned}
$$

where $\sup_{x\in\Omega} |b(x,\cdot)|_{\alpha,\mathbb{R}} < \infty$, $u_0 \in C^\alpha(\overline{\Omega})$, $\partial\Omega \in C^{2+\alpha}$.

1. Establish the existence and uniqueness for a classical solution globally in time (i.e., for the time interval $[0,\infty)$).
2. If $b(x,t) \geqslant 0$, prove a comparison principle for this system, i.e., if $u_1, u_2 \in C(\overline{\Omega}_T) \cap C^2(\Omega_T)$ satisfy

$$
\begin{aligned}
(u_1)_t - \Delta u_1 &\geqslant \int_\Omega b(y,t)u_1(y,t)dy && \text{in } \Omega \times (0,T), \\
(u_2)_t - \Delta u_2 &\leqslant \int_\Omega b(y,t)u_2(y,t)dy && \text{in } \Omega \times (0,T), \\
u_1 &\geqslant u_2 && \text{on } \partial\Omega \times (0,T), \\
u_1(x,0) &\geqslant u_2(x,0) && \text{in } \Omega,
\end{aligned}
$$

   then $u_1 \geqslant u_2$ in Ω_T.
3. Give an example showing that in general the comparison principle is not valid if the assumption "$b \geqslant 0$" is dropped.

Chapter 5
Finite Time Blow-Up for Evolution Equations

As indicated in the preface, we will emphasize the method and techniques for studying blow-up problems. While many of these methods apply to more general equations, we shall use the simplest model in our lectures to avoid lengthy computations.

The equation

$$u_t - \Delta u = f(u) \qquad \text{in } \Omega_T \equiv \Omega \times (0,T), \tag{5.1}$$

$$u = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{5.2}$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in \Omega \tag{5.3}$$

can be used to model solid fuel ignition (see [15] for details). The function $f(u)$ is typically a nonlinear function such as u^p $(p > 1)$, $\exp(u)$, etc. If the source term is on the boundary, we then have the following system:

$$u_t - \Delta u = 0 \qquad \text{in } \Omega_T \equiv \Omega \times (0,T), \tag{5.4}$$

$$\frac{\partial u}{\partial n} = f(u) \qquad \text{on } \partial\Omega \times (0,T), \tag{5.5}$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in \Omega, \tag{5.6}$$

where $\mathbf{n}$ is the unit exterior normal vector.

The first question is the existence and uniqueness for either (5.1)–(5.3) or (5.4)–(5.6). For the corresponding linear problem, the existence and uniqueness is stated in Theorem 3.5 for the Dirichlet problem. The Neumann problem can be studied in a similar manner. For the nonlinear problem, the questions about existence (locally in time) and uniqueness have been answered in a similar manner (see Exercise 4.1).

One can, of course, study the problem of combined heat source in the interior and on the boundary, the problem of systems of more than one equation, etc. Many results have been obtained in recent decades on these types of problems.

In this chapter, we want study whether a solution exists globally in time, or there is a finite time blow-up. If the blow-up occurs, we want to find out the blow-up rate and the asymptotic behavior near the blow-up point.

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Definition 5.1 (L^∞ blow-up). We say that a solution u blows up (or thermal runaway) at $t = T$, if there exists (x_n, t_n), $t_n \nearrow T$, such that $|u(x_n, t_n)| \to +\infty$. In this case, we say that the solution blows up in finite time if T is finite; if there exists a sequence $\{y_n\}$ and $t_n \nearrow T$ such that $y_n \to x$ and $|u(y_n, t_n)| \to +\infty$, then we say that x is a blow-up point. The collection of all blow up points is called the blow-up set.

For the finite time blow-up, Osgood [113] gave a criterion, namely the right-hand side nonlinear term must satisfy

$$\int^\infty \frac{ds}{f(s)} < \infty.$$

The earliest blow-up results on parabolic equations are due to Kaplan [80] and Fujita [46]. Some early papers appeared in the 1970s: Tsutsumi [131], Hayakawa [70], Levine [85], Levine–Payne [90, 91], Walter [133], Ball [7], Kobayashi–Siaro–Tanaka [82], Aronson–Weinberger [6]. We begin by presenting some of their results.

5.1 Finite Time Blow-Up: Kaplan's First Eigenvalue Method

If we drop the diffusion term in (5.1), the *positive solution* of the ordinary differential equation (ODE) $u_t = f(u)$ will blow-up in finite time for any positive initial data, provided f is defined for all $u \in \mathbb{R}$, and satisfies

$$f(u) > 0 \quad \text{for } u > 0, \qquad \int_M^\infty \frac{du}{f(u)} < \infty, \tag{5.7}$$

for some $M > 0$. So a natural question is whether the diffusion is strong enough to diffuse the *energy* to prevent a finite time blow-up.

Remark 5.1 (necessary condition). (5.7) is a *necessary condition* for blow-up to occur. In fact, if $\displaystyle\int_M^\infty \frac{du}{f(u)} = \infty$, one can then obtain global existence of (5.1)–(5.3) by comparing its solution with an ODE solution.

Here we shall introduce the first eigenvalue method introduced in 1963 by Kaplan [80]. As we shall see from the proof, it is a very simple method and yet applies to a large class of equations.

Theorem 5.1. *Let Ω be a bounded domain with $\partial\Omega \in C^1$. Assume that f is convex (i.e., $f'' \geqslant 0$), and (5.7) is satisfied. Let $u \in C(\overline{\Omega}_T) \cap C^2(\Omega_T)$ be a solution of (5.1)–(5.3). If $\displaystyle\int_\Omega u_0(x)dx$ is sufficiently large, then the solution u must blow-up in finite time.*

Proof. Consider the eigenvalue problem

$$-\Delta\phi = \lambda_1\phi \quad \text{in } \Omega, \tag{5.8}$$

$$\phi = 0 \quad \text{on } \partial\Omega, \tag{5.9}$$

$$\int_\Omega \phi(x)dx = 1, \tag{5.10}$$

where λ_1 is the first eigenvalue, given by

$$\lambda_1 = \inf\{\|\nabla u\|_{L^2(\Omega)}; u \in K\}, \qquad K = \{u \in H_0^1(\Omega); \quad \|u\|_{L^2(\Omega)} = 1\}.$$

The system (5.8)–(5.10) has a solution $\phi \in H_0^1(\Omega)$ such that $\phi(x) > 0$ in Ω (see Exercise 5.1). Let

$$G(t) = \int_\Omega u(x,t)\phi(x)dx.$$

It is clear that $G(t)$ is well defined on the existence interval of the solution u. Using (5.1) and (5.2), we find that

$$\begin{aligned} G'(t) &= \int_\Omega u_t(x,t)\phi(x)dx \\ &= \int_\Omega u(x,t)\Delta\phi(x)dx + \int_\Omega f(u(x,t))\phi(x)dx \\ &= -\lambda_1 G(t) + \int_\Omega f(u(x,t))\phi(x)dx. \end{aligned} \tag{5.11}$$

By Jensen's inequality,

$$\int_\Omega f(u(x,t))\phi(x)dx \geqslant f\Big(\int_\Omega u(x,t)\phi(x)dx\Big) = f(G(t)).$$

Substituting this into (5.11), we obtain

$$G'(t) \geqslant -\lambda_1 G(t) + f(G(t)). \tag{5.12}$$

If u remains finite for all t, then $G(t)$ is defined for all t. However, from the ODE theory, $G(t)$ will blow-up in finite time under the given assumptions, provided $G(0)$ is large enough. □

The assumption that the initial datum is "large" cannot be dropped. The solution can exist globally in time if this assumption is dropped. We use $f(u) = u^p$ to illustrate this in the following theorem.

We take ϕ to be the solution of (5.8)–(5.10) and take $\psi(x) = \eta\phi(x)$. If $0 < \eta \ll 1$, then $-\Delta\psi - \psi^p = \eta(\lambda_1\phi - \eta^{p-1}\phi^p) \geqslant 0$. It follows that ψ is a supersolution and $u(x,t) \leqslant \psi(x)$ for all t if initially $0 \leqslant u_0(x) \leqslant \psi(x)$. We proved

Theorem 5.2 (Global existence). *In the case $f(u) = u^p$, if $0 \leqslant u_0(x) \leqslant \psi(x)$, then the solution exists globally in time.*

5.2 Finite Time Blow-Up: Concavity Method

In this section we introduce another method to establish finite time blow-up. This method, introduce by Levine–Payne in the papers [90,91] and Levine [85] in 1970s, uses the concavity of an auxiliary function.

For simplicity, we shall only consider (5.1)–(5.3) with

$$f(u) = |u|^{p-1}u \qquad (p > 1). \tag{5.13}$$

This concavity method is actually powerful enough to be applied to many other types of second parabolic equations as well as other types of evolution equations. It makes no use of maximum principles and the following theorem is only a special case discussed in [85].

Theorem 5.3 (Levine). *Let Ω be a smooth domain. If f is given by (5.13) and $u_0(x)$ satisfies*

$$-\frac{1}{2}\int_\Omega |\nabla u_0(x)|^2 dx + \frac{1}{p+1}\int_\Omega |u_0(x)|^{p+1} dx > 0, \tag{5.14}$$

then the solution of (5.1)–(5.3) must blow-up in finite time.

Proof. Multiplying the equation by u and u_t, respectively, and then integrating over Ω, we obtain

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega u^2 dx\Big) + \int_\Omega |\nabla u|^2 dx = \int_\Omega |u|^{p+1} dx, \tag{5.15}$$

$$\int_\Omega u_t^2 dx = -\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega |\nabla u|^2 dx\Big) + \frac{d}{dt}\Big(\frac{1}{p+1}\int_\Omega |u|^{p+1} dx\Big). \tag{5.16}$$

Let

$$J(t) = -\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{p+1}\int_\Omega |u|^{p+1} dx.$$

Then (5.16) gives

$$J'(t) = \int_\Omega u_t^2 dx \ge 0, \tag{5.17}$$

which implies that

$$J(t) = J(0) + \int_0^t \int_\Omega u_t^2 dx dt. \tag{5.18}$$

Introduce a new function

$$I(t) = \int_0^t \int_\Omega u^2 dx dt + A,$$

where $A > 0$ is to be determined. Then

$$I'(t) = \int_\Omega u^2 dx,$$

and

$$I''(t) = \frac{d}{dt}\int_\Omega u^2 dx = -2\int_\Omega |\nabla u|^2 dx + 2\int_\Omega |u|^{p+1} dx, \tag{5.19}$$

where we used (5.15) in deriving the second equality.

By comparing the terms in $J(t)$ and (5.19) we have, for $\delta = \frac{1}{2}(p-1) > 0$,

$$I''(t) \geq 4(1+\delta)J(t) = 4(1+\delta)\Big(J(0) + \int_0^t \int_\Omega u_t^2 dx dt\Big), \tag{5.20}$$

where the last equality is from (5.18). Clearly

$$I'(t) = \int_\Omega u^2 dx = 2\int_0^t \int_\Omega u u_t dx dt + \int_\Omega u_0^2 dx. \tag{5.21}$$

It follows that, for any $\varepsilon > 0$,

$$I'(t)^2 \leq 4(1+\varepsilon)\int_0^t \int_\Omega u^2 dx dt \int_0^t \int_\Omega u_t^2 dx dt + \Big(1 + \frac{1}{\varepsilon}\Big)\Big[\int_\Omega u_0^2 dx\Big]^2. \tag{5.22}$$

Combining the above estimates, we find that for $\alpha > 0$,

$$\begin{aligned} &I''(t)I(t) - (1+\alpha)I'(t)^2 \\ &\geq 4(1+\delta)\Big[J(0) + \int_0^t \int_\Omega u_t^2 dx dt\Big]\Big[\int_0^t \int_\Omega u^2 dx dt + A\Big] \\ &\quad -(1+\alpha)\Big[4(1+\varepsilon)\int_0^t \int_\Omega u^2 dx dt \int_0^t \int_\Omega u_t^2 dx dt\Big] \\ &\quad -(1+\alpha)\Big(1+\frac{1}{\varepsilon}\Big)\Big[\int_\Omega u_0^2 dx\Big]^2. \end{aligned}$$

Now we choose ε and α small enough such that

$$1 + \delta \geq (1+\alpha)(1+\varepsilon). \tag{5.23}$$

By our assumption, $J(0) > 0$. Thus we can choose A to be large enough so that

$$I''(t)I(t) - (1+\alpha)I'(t)^2 > 0. \tag{5.24}$$

The inequality (5.24) implies that

$$\frac{d}{dt}\Big(\frac{I'(t)}{I^{\alpha+1}(t)}\Big) > 0,$$

so that

$$\frac{I'(t)}{I^{\alpha+1}(t)} \geqslant \frac{I'(0)}{I^{\alpha+1}(0)} \quad \text{for } t > 0.$$

It follows that $I(t) = \int_0^t \int_\Omega u^2 dx dt + A$ cannot remain finite for all t. □

Remark 5.2. The domain Ω is can be either bounded or unbounded.

Remark 5.3. The condition (5.14) is explicit on the initial datum. Such u_0 always exists. One can pick any non-trivial function $\psi \in H^1(\Omega) \cap L^{p+1}(\Omega)$ and let $u_0(x) = \lambda\psi(x)$, $\lambda \gg 1$.

Remark 5.4. Note that if the solution of (5.1)–(5.3) (with $f(u) = |u|^{p-1}u$) is global, then we must have

$$-\frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 dx + \frac{1}{p+1}\int_\Omega |u(x,t)|^{p+1} dx < 0 \quad \text{for all } t > 0,$$

this estimate is useful in establish a global bound for global solutions.

5.3 Finite Time Blow-Up: A Comparison Method

If the system under consideration has a comparison principle, then the solution must blow up if a subsolution blows up in finite time. There are no general rules, however, on how to construct comparison functions. Here we give one example.

Theorem 5.4. *Consider the system (5.4) and (5.6) in a bounded smooth domain with $f(u) = u^p$ ($p > 1$) and $u_0(x) \geqslant 0$. In this case all nontrivial solutions blow up in finite time.*

Proof. The result can be established by several methods (c.f. [33]). The proof here is very simple and is taken from [75]. By the maximum principle (Theorems 3.6 and 3.7), $\inf_{x\in\Omega} u(x,\varepsilon) > 0$ (for small $\varepsilon > 0$). Replacing $t = 0$ by $t = \varepsilon$ if necessary, we may assume without loss of generality that $\inf_{x\in\Omega} u_0(x) = c > 0$. Take $\phi(x,t)$ (the existence is guaranteed by Exercise 4.1(1)) such that

$$\phi_t - \Delta\phi = 0 \quad \text{for } x \in \Omega,\ t > 0,$$
$$\frac{\partial\phi}{\partial n} = \phi^p \quad \text{for } x \in \partial\Omega,\ t > 0$$
$$\phi(x,0) = c \quad \text{for } x \in \Omega.$$

By comparison principle (Exercise 4.1(3)),

$$u(x,t) \geqslant \phi(x,t) \quad \text{for } x \in \Omega,\ t > 0$$

as long as both solutions exist. Thus the problem reduces to proving $\phi(x,t)$ blows up in finite time.

By the maximum principle, $\phi(x,t) \geqslant c$ for all $t > 0$. Thus for any $\eta > 0$, the function $\psi(x,t) = \phi(x,t+\eta) - \phi(x,t)$ satisfies:

$$\begin{aligned} &\psi_t - \Delta\psi = 0 \quad \text{for } x \in \Omega,\ t > 0, \\ &\frac{\partial\psi}{\partial n} = p\xi^{p-1}(x,t)\psi \quad \text{for } x \in \partial\Omega,\ t > 0, \\ &\psi(x,0) = \phi(x,\eta) - c \geqslant 0 \quad \text{for } x \in \Omega, \end{aligned} \tag{5.25}$$

where $\xi(x,t)$ is a number between $\phi(x,t)$ and $\phi(x,t+\eta)$. One can now easily derive that $\psi(x,t) \geqslant 0$, which implies that $\phi_t(x,t) \geqslant 0$ for $x \in \overline{\Omega}, t > 0$. In particular, applying the maximum principle (Theorems 3.6 and 3.7) to u_t, we obtain

$$\inf_{x\in\Omega} \phi_t(x,\varepsilon) > 0 \text{ for small } \varepsilon > 0.$$

Let $w(x,t) = \phi_t(x,t) - \delta\phi^p(x,t)$. Then a direct calculation shows that

$$\frac{\partial w}{\partial t} - \Delta w \geq 0 \quad \text{for } x \in \Omega,\ t > \varepsilon,$$
$$\frac{\partial w}{\partial n} = p\phi^{p-1}w \quad \text{for } x \in \partial\Omega,\ t > \varepsilon,$$

and $w(x,\varepsilon) \geq 0$ if δ is small enough. It follows that $w(x,t) \geq 0$ for $t \geqslant \varepsilon$, which implies that $\phi_t \geq \delta\phi^p$ for $t \geqslant \varepsilon$. Thus $\phi(x,t)$ blows up in finite time, and $u(x,t)$ must also blow up at a (perhaps different) finite time. □

5.4 Fujita Types of Results on Unbounded Domains

One of the earliest results is Fujita's critical exponent (Fujita [46]) on the simple system

$$u_t - \Delta u = u^p, \quad x \in \mathbb{R}^n,\ t > 0, \qquad (p > 1) \tag{5.26}$$

$$u(x,0) = u_0(x) \geqslant 0, \quad x \in \mathbb{R}^n. \tag{5.27}$$

We can compare the solution with a blow-up solution of (5.1)–(5.3) (with $f(u) = u^p$), so that the solution of (5.26) and (5.27) always blows up in finite time if the initial datum $u_0(x)$ is large enough in a certain sense. The question is whether or not there are global solutions for all time.

If u is *small*, then u^p is *very small* if p is large. As Fujita observed, if p is *large* and u_0 is small, then there exist global solutions. On the other hand, if p is close to 1, then all positive solutions blow-up in finite time. In the case of the nonlinearity given by u^p on the whole space, the cutoff value, or the *critical exponent*, is $1 + \frac{2}{n}$. Actually Fujita's result does not include the critical exponent itself. The critical exponent itself actually belongs to the blow-up case (see Hayakawa [70], Kobayashi–Kobayashi–Siaro–Tanaka [82], Aronson–Weinberger [6], Weissler [134]).

The proof given here uses the representation of the solution in an integral equation in terms of its fundamental solution. This approach does not need a separate proof to include the critical exponent itself in the blow-up case.

The solution of (5.26) and (5.27) has an integral representation [30, p. 51, (17)]:

$$u(x,t) = \int_{\mathbb{R}^n} \Gamma(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^n} \Gamma(x-y,t-\tau)u^p(y,\tau)dyd\tau, \quad (5.28)$$

where

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp\Big(-\frac{|x|^2}{4t}\Big)$$

satisfies

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)\Gamma(x,t) = 0 \quad \text{for } (x,t) \neq (0,0).$$

Theorem 5.5. *(i) If $p > 1 + \frac{2}{n}$, then the solution of (5.26) and (5.27) is global in time, provided the initial datum satisfies, for some small $\varepsilon > 0$,*

$$u_0(x) \leqslant \varepsilon \Gamma(x,1) \quad \textit{for } x \in \mathbb{R}^n.$$

(ii) If $p \leqslant 1 + \frac{2}{n}$, then all nontrivial solutions of (5.26) and (5.27) blow-up in finite time.

Proof. To prove (i), we construct a global supersolution. Since $p > 1 + \frac{2}{n}$, we can take $\eta > 0$ to be small enough such that

$$p > 1 + \frac{2}{n} + \frac{2}{n}(p-1)\eta,$$

or equivalently,

$$1 + \frac{n}{2} - \eta < p\Big(\frac{n}{2} - \eta\Big).$$

Let $\psi(x,t) = \lambda t^{\eta}\Gamma(x,t)$, then for $\lambda > 0$ to be small enough, we have

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)\psi - \psi^p &= \lambda\eta t^{\eta-1}\Gamma(x,t) - \lambda^p t^{\eta p}\Gamma^p(x,t)\\ &= \frac{\lambda\eta}{(4\pi)^{n/2}} t^{-(1+n/2-\eta)} \exp\Big(-\frac{|x|^2}{4t}\Big)\\ &\quad -\frac{\lambda^p}{(4\pi)^{np/2}} t^{-p(n/2-\eta)} \exp\Big(-\frac{p|x|^2}{4t}\Big)\\ &> 0 \quad \text{for } t \geqslant 1.\end{aligned}$$

It follows that $u(x,t) \leqslant \psi(x,t+1)$ for all $t > 0$ if $u_0(x) \leqslant \psi(x,1)$.

To prove (ii), we use the integral representation (5.28). We assume that the solution remains finite for all finite t and want to derive a contradiction. We write $u(x,t) = u_1(x,t) + u_2(x,t)$, where

$$\begin{aligned}u_1 &= \int_{\mathbb{R}^n} \Gamma(x-y,t)u_0(y)dy,\\ u_2 &= \int_0^t \int_{\mathbb{R}^n} \Gamma(x-y,t-\tau)u^p(y,\tau)dyd\tau.\end{aligned}$$

Replacing $t = 0$ by $t = \varepsilon > 0$ if necessary, we may assume without loss of generality that $u_0(y) \geqslant c_0 > 0$ for $|y| < 1$ (by applying Theorem 3.7).

A direct computation shows that

$$\begin{aligned}u_1(x,t) &\geqslant \frac{c_0}{(2\pi t)^{n/2}} \int_{B_1(0)} \exp\Big(-\frac{|x|^2+|y|^2}{2t}\Big)dy\\ &= \frac{c_0}{(2\pi)^{n/2}} \exp\Big(-\frac{|x|^2}{2t}\Big) \int_{|\xi|\leqslant 1/\sqrt{t}} \exp\Big(-\frac{|\xi|^2}{2}\Big)d\xi \qquad (5.29)\\ &\geqslant \frac{c_1}{(2\pi t)^{n/2}} \exp\Big(-\frac{|x|^2}{2t}\Big) \quad \text{for } t > 1\end{aligned}$$

for some $c_1 > 0$.

Since $\displaystyle\int_{\mathbb{R}^n} \Gamma(x-y,t-\tau)dy = 1$ for any x and $t > \tau$, we can apply Jensen's inequality to obtain

$$u_2(x,t) \geqslant \int_0^t \Big(\int_{\mathbb{R}^n} \Gamma(x-y,t-\tau)u(y,\tau)dy\Big)^p d\tau.$$

Let

$$G(t) = \int_{\mathbb{R}^n} u(x,t)\Gamma(x,t)dx.$$

Then, for $t > 1$,

$$\begin{aligned} G(t) &= \int_{\mathbb{R}^n} u_1(x,t)\Gamma(x,t)dx + \int_{\mathbb{R}^n} u_2(x,t)\Gamma(x,t)dx \\ &\geqslant \frac{c_2}{t^{n/2}} + \int_0^t \Big[\int_{\mathbb{R}^n} \Gamma(x,t)\Big(\int_{\mathbb{R}^n} \Gamma(x-y,t-\tau)u(y,\tau)dy\Big)^p dx\Big] d\tau \\ &\geqslant \frac{c_2}{t^{n/2}} + \int_0^t \Big[\int_{\mathbb{R}^n} \Gamma(x,t)\Big(\int_{\mathbb{R}^n} \Gamma(x-y,t-\tau)u(y,\tau)dy\Big) dx\Big]^p d\tau \\ &\qquad \text{(by Jensen's inequality)} \\ &= \frac{c_2}{t^{n/2}} + \int_0^t \Big\{\int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^n} \Gamma(x,t)\Gamma(x-y,t-\tau)dx\Big] u(y,\tau)dy\Big\}^p d\tau. \end{aligned} \tag{5.30}$$

It is clear that

$$\begin{aligned} &\int_{\mathbb{R}^n} \Gamma(x,t)\Gamma(x-y,t-\tau)dx \\ &\quad = \frac{1}{(4\pi t)^{n/2}[4\pi(t-\tau)]^{n/2}} \int_{\mathbb{R}^n} \exp\Big(-\frac{|x|^2}{4t} - \frac{|x-y|^2}{4(t-\tau)}\Big)dx \\ &\quad = \Gamma(y,\tau)\frac{(4\pi\tau)^{n/2}}{(4\pi t)^{n/2}[4\pi(t-\tau)]^{n/2}} \int_{\mathbb{R}^n} \exp\Big(\frac{|y|^2}{4\tau} - \frac{|x|^2}{4t} - \frac{|x-y|^2}{4(t-\tau)}\Big)dx. \end{aligned}$$

Since

$$\begin{aligned} &\frac{|y|^2}{4\tau} - \frac{|x|^2}{4t} - \frac{|x-y|^2}{4(t-\tau)} \\ &\qquad \geq \frac{|y|^2}{4\tau} - \frac{|x-y|^2 + |y|^2 + 2|x-y||y|}{4t} - \frac{|x-y|^2}{4(t-\tau)} \\ &\qquad = \frac{1}{4t}\Big(-2|x-y||y| + \frac{t-\tau}{\tau}|y|^2\Big) - \frac{|x-y|^2}{4(t-\tau)} - \frac{|x-y|^2}{4t} \\ &\qquad \geq -\frac{\tau|x-y|^2}{4t(t-\tau)} - \frac{|x-y|^2}{4(t-\tau)} - \frac{|x-y|^2}{4t} \\ &\qquad \geq -\frac{3|x-y|^2}{4(t-\tau)} \quad \text{for } 0 < \tau < t, \end{aligned}$$

we derive

$$\int_{\mathbb{R}^n} \exp\Big(\frac{|y|^2}{4\tau} - \frac{|x|^2}{4t} - \frac{|x-y|^2}{4(t-\tau)}\Big)dx \geqslant \int_{\mathbb{R}^n} \exp\Big(-\frac{3|x-y|^2}{4(t-\tau)}\Big)dx = c_3(t-\tau)^{n/2}$$

for $0 < \tau < t$ ($c_3 > 0$). Substituting this estimate into (5.30), we obtain, for some $c_2 > 0, c_3 > 0$,

$$G(t) \geqslant \frac{c_2}{t^{n/2}} + c_3 \int_0^t \Big(\frac{\tau^{n/2}}{t^{n/2}}\Big)^p G^p(\tau)d\tau \quad \text{for } t > 1.$$

We can rewrite this inequality as

$$t^{np/2}G(t) \geqslant c_2 t^{n(p-1)/2} + c_3 \int_0^t \tau^{np/2} G^p(\tau)d\tau \quad \text{for } t > 1. \tag{5.31}$$

Denote the right-hand side of the above inequality by $g(t)$, then for $t > 1$,

$$g(t) \geqslant c_2 t^{n(p-1)/2}, \tag{5.32}$$

$$g'(t) \geqslant c_3 t^{np/2} G^p(t) \geqslant c_3 t^{np/2} \Big(\frac{1}{t^{np/2}} g(t)\Big)^p = c_3 t^{(1-p)np/2} g^p(t), \tag{5.33}$$

which implies

$$\frac{c_2^{1-p}}{p-1} t^{-n(p-1)^2/2} \geqslant \frac{1}{p-1} g^{1-p}(t) > c_3 \int_t^T \tau^{-(p-1)np/2} d\tau \quad \text{for } T > t \geqslant 1. \tag{5.34}$$

The right-hand side of the above inequality is unbounded as $T \to \infty$ if $p \leqslant 1 + \frac{2}{np}$, which gives a contradiction in this case. In the case $1 + \frac{2}{np} < p < 1 + \frac{2}{n}$, we have $\frac{n(p-1)^2}{2} > -1 + \frac{(p-1)np}{2}$, so we get a contradiction by letting $T \to \infty$ and then taking $t \gg 1$.

Finally, in the case $p = 1 + \frac{2}{n}$, we derive from (5.29), for $t > 1$,

$$u^p(x,t) \geqslant u_1^p(x,t) \geqslant \frac{c_1^p}{(2\pi t)^{np/2}} \exp\Big(-\frac{p|x|^2}{2t}\Big). \tag{5.35}$$

Substituting this estimate into the expression for u_2, we obtain, for $t > 2$,

$$\begin{aligned}
u(x,t) &\geqslant u_2(x,t) \\
&\geqslant \int_1^t \int_{\mathbb{R}^n} \Gamma(x-y, t-\tau) \frac{c_1^p}{(2\pi\tau)^{np/2}} \exp\Big(-\frac{p|y|^2}{2\tau}\Big) dy d\tau \\
&\geqslant \frac{c_4}{t^{n/2}} \exp\Big(-\frac{|x|^2}{t}\Big) \int_1^{t/2} \frac{t^{n/2}}{\tau^{1+n/2}(t-\tau)^{n/2}} \\
&\quad \times \Big\{ \int_{\mathbb{R}^n} \exp\Big(\frac{|x|^2}{t} - \frac{|x|^2 + |y|^2}{2(t-\tau)} - \frac{p|y|^2}{2\tau}\Big) dy \Big\} d\tau \\
&\geqslant \frac{c_4}{t^{n/2}} \exp\Big(-\frac{|x|^2}{t}\Big) \int_1^{t/2} \frac{t^{n/2}}{\tau^{1+n/2}(t-\tau)^{n/2}} \\
&\quad \times \Big\{ \int_{\mathbb{R}^n} \exp\Big(\frac{|x|^2}{t} - \frac{|x|^2 + |y|^2}{t} - \frac{p|y|^2}{2\tau}\Big) dy \Big\} d\tau \\
&\geqslant \frac{c_5}{t^{n/2}} \exp\Big(-\frac{|x|^2}{t}\Big) \int_1^{t/2} \frac{d\tau}{\tau} \\
&= \frac{c_5}{t^{n/2}} \exp\Big(-\frac{|x|^2}{t}\Big) \log(t/2).
\end{aligned} \tag{5.36}$$

(Please note the extra $\log(t/2)$ function in the above inequality!) Hence, for $t > 2$,

$$G(t) \geqslant \int_{\mathbb{R}^n} \Gamma(x,t) \frac{c_5}{t^{n/2}} \exp\Big(- \frac{|x|^2}{t}\Big) \log(t/2) dx \geqslant \frac{c_6}{t^{n/2}} \log t. \tag{5.37}$$

Using this estimate, we now derive from (5.31) (recall that $p = 1 + \frac{2}{n}$), for $t > 2$,

$$t^{np/2}G(t) = \frac{1}{2}t^{np/2}G(t) + \frac{1}{2}t^{np/2}G(t) \geqslant c_7 t \log t + \frac{c_3}{2}\int_0^t \tau^{np/2}G^p(\tau)d\tau. \tag{5.38}$$

Denoting the right-hand side of the above inequality by $g(t)$ and following the same arguments as in (5.32)–(5.34), we derive a contradiction. □

Remark 5.5. There are numerous works regarding blow-up phenomena. There are numerous works regarding the critical exponents for various types of domains, various equations, coupled system of equations, the equations with heat source on the boundaries, and the coupled systems of equations with boundary heat sources.

Meijer [104] considered the case where the nonlinearity is given by u^p for a parabolic equation with variable coefficients in divergence form. The Fujita's critical exponents for a general domain is established. The exponent was found explicitly for other types of domains by Meijer [103], Levine–Meijer [89], Bundle–Levine [11, 12], and a formula was found for products of domains by Ohta–Kaneko [112]. Interesting results on oscillating solutions were also obtained in Mizoguchi–Yanagida [107, 108]. The Fujita's critical exponents are also studied on a manifold by Zhang [142–144].

There are also many works that deal with the situation when u^p is replaced by other nonlinearities such as $a(x)u^p$, or $t^k|x|^\sigma u^p$ (cf. Hamada [69], Levine–Meijer [88, 89], Pinsky [116, 117], etc).

There is also a flurry of works on equations with nonlinear principle parts such as $u_t - \Delta u^{1+\sigma}$ and $u_t - \nabla(|\nabla u|^\sigma \nabla u^m)$ (see, for example, the papers Galaktionov [47, 49], Galaktionov–Levine [53], Mochizuki–Mukai [110], Mochizuki–Suzuki [111], Souplet–Weissler [127], Qi [119, 120], Suzuki [129, 130]).

The blow-up phenomenon and the Fujita's critical exponents types of results were also studied for equations with boundary source terms. In the one-space-dimensional case, Galaktionov–Levine [52] established such results for several types of systems. For the half-space and principle part $u_t - \Delta u$, Deng–Fila–Levine [27] established the n-space-dimensional case (see Theorem 5.6). Hu–Yin [76, 77] extended this case to convex (unbounded) domains in $\mathbb{R}^n$, and to sectorial domains with mixed Dirichlet and nonlinear Neumann heat flux. Amann–Fila [5] studied critical exponents in the case of Laplacian with dynamical boundary conditions. The effect of convection is studied in [3]. For more details, see the surveys [28, 34, 86] and the references therein.

Remark 5.6. Some of the techniques discussed here can also be applied to systems of equations such as

$$\begin{aligned} u_t - \Delta u &= f(u,v) \quad \text{in } \Omega \times (0,T), \\ v_t - \Delta v &= g(u,v) \quad \text{in } \Omega \times (0,T), \end{aligned} \tag{5.39}$$

with appropriate boundary and initial conditions. The coupling can also appear on the boundary:

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } \Omega \times (0,T), \\ v_t - \Delta v &= 0 \quad \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} &= f(u,v) \quad \text{in } \partial\Omega \times (0,T), \\ \frac{\partial u}{\partial n} &= g(u,v) \quad \text{in } \partial\Omega \times (0,T). \end{aligned} \tag{5.40}$$

One can also consider the combined interior and boundary source terms, or a more general principle part such as $u_t - \Delta u^{1+\sigma}$ or $u_t - \nabla(|\nabla u|^\sigma \nabla u^m)$. One can study the Fujita's critical exponents, blow-up rate (to be discussed in Chap. 7), and general properties of these systems. For Fujita's critical exponents type of results, we refer the readers to the papers [31, 32, 36, 37, 78, 87, 99, 109, 121, 124, 132, 147] and the references therein.

5.5 Exercises

5.1. Prove that (5.8)–(5.10) has a positive solution $u \in H^1(\Omega)$. Show that this solution is C^∞ in the interior, and $C^{k+\beta}$ ($k \geqslant 2$) on the boundary, if $\partial\Omega \in C^{k+\beta}$.

5.2. Give appropriate assumptions on $\partial\Omega$ and $f(u)$ and prove a finite time blow-up result for the system (5.4)–(5.6) using concavity method.

5.3. (See Kalantarov–Ladyzhenskaya [81]) Suppose that

$$I''(t)I(t) - (1+\alpha)I'(t)^2 > -2C_1 I(t)I'(t) - C_2 I^2(t), \quad t > 0$$

for some $C_1, C_2 > 0$. Suppose that

$$I(0) > 0, \quad I'(0) + \frac{-C_1 - \sqrt{C_1^2 + \alpha C_2}}{\alpha} I(0) > 0.$$

Prove that $I(t)$ must blow up in finite time.

5.4. For the equation on the half-space with a nonlinear heat source:

$$u_t - \Delta u = 0, \quad x \in \mathbb{R}^n_+,\ t > 0, \qquad (p > 1) \tag{5.41}$$

$$\frac{\partial u}{\partial n} = u^p \geqslant 0, \quad x \in \partial\mathbb{R}^n_+, \tag{5.42}$$

$$u(x,0) = u_0(x) \geqslant 0, \tag{5.43}$$

where $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n), x_1 > 0\}$, prove the following theorem using the method introduced in Sect. 5.4.

Theorem 5.6 (see Deng–Fila–Levine [27]). *(i) If $p > 1 + \frac{1}{n}$, then the solution of (5.41)–(5.43) is global in time, if the initial datum satisfies, for some small $\varepsilon > 0$,*

$$u_0(x) \leqslant \varepsilon \Gamma(x,1) \quad \textit{for } x \in \mathbb{R}^n_+.$$

(ii) If $p \leqslant 1 + \frac{1}{n}$, then all nontrivial solutions of (5.41)–(5.43) blow-up in finite time.

Chapter 6
Steady-State Solutions

When the solutions of (5.1)–(5.3), or (5.4)–(5.6) are independent of t, they are called steady-state solutions, or stationary solutions. These solutions are the possible limits as $t \to \infty$ of the corresponding time-dependent solutions if the time-dependent solution is global. There is no panacea for their study. However, many methods were developed to study these types of systems.

6.1 Existence: Upper and Lower Solution Methods

Consider the Dirichlet problem:

$$-\Delta u = f(x,u) \quad \text{in } \Omega, \tag{6.1}$$
$$u = g \quad \text{on } \partial\Omega. \tag{6.2}$$

Definition 6.1. A function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is said to be a lower (upper) solution if

$$-\Delta u \leqslant (\geqslant)\ f(x,u) \quad \text{in } \Omega,$$
$$u \leqslant (\geqslant)\ g \quad \text{on } \partial\Omega.$$

Theorem 6.1. *Let Ω be a bounded domain with $\partial\Omega \in C^{2+\alpha}$. Assume that $f \in C^\alpha(\overline{\Omega} \times \mathbb{R})$, $f_u^- \in L^\infty(\Omega \times \mathbb{R})$ and $g \in C(\overline{\Omega})$. If the system has an upper solution $\overline{u}$ and a lower solution $\underline{u}$ such that $\underline{u}(x) \leqslant \overline{u}(x)$ on $\overline{\Omega}$, then (6.1) and (6.2) has a solution $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that $\underline{u}(x) \leqslant u(x) \leqslant \overline{u}(x)$ on $\overline{\Omega}$.*

Proof. Let $\underline{u}_0(x) = \underline{u}(x)$ and $\overline{u}_0(x) = \overline{u}(x)$. Define $c = \sup_{\Omega \times \mathbb{R}} f_u^-(x,u)$. Then the function $F(x,u) = cu + f(x,u)$ satisfies

$$F_u(x,u) \geqslant 0.$$

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For $n \geqslant 1$ we define $\underline{u}_n(x)$ (the existence and uniqueness is given by Remark 2.3) to be the solution of

$$-\Delta \underline{u}_n + c\underline{u}_n = F(x, \underline{u}_{n-1}(x)) \quad \text{in } \Omega,$$
$$\underline{u}_n = g \quad \text{on } \partial\Omega,$$

and $\overline{u}_n(x)$ to be the solution of

$$-\Delta \overline{u}_n + c\overline{u}_n = F(x, \overline{u}_{n-1}(x)) \quad \text{in } \Omega,$$
$$\overline{u}_n = g \quad \text{on } \partial\Omega.$$

From the linear theory (using a barrier function on the boundary and the Schauder theory in the interior, see Remark 2.3), $\underline{u}_n, \overline{u}_n \in C(\overline{\Omega}) \cap C^2(\Omega)$. We want to prove by induction that (1) $\underline{u}_{n-1} \leqslant \underline{u}_n$, (2) $\underline{u}_n \leqslant \overline{u}_n$, (3) $\overline{u}_n \leqslant \overline{u}_{n-1}$.

For $n = 1$, we use the definition of $\underline{u}_0$ and $\underline{u}_1$ to derive

$$-\Delta(\underline{u}_0 - \underline{u}_1) + c(\underline{u}_0 - \underline{u}_1) \leqslant F(x, \underline{u}_0(x)) - F(x, \underline{u}_0(x)) = 0 \quad \text{in } \Omega,$$
$$\underline{u}_0 - \underline{u}_1 \leqslant 0 \quad \text{on } \partial\Omega.$$

It follows from the comparison principle (Theorem 2.1) that $\underline{u}_0 - \underline{u}_1 \leqslant 0$. We now inductively assume $\underline{u}_{n-1} \leqslant \underline{u}_n$ $(n \geqslant 1)$, then by monotonicity $F(x, \underline{u}_{n-1}(x)) \leqslant F(x, \underline{u}_n(x))$. Therefore

$$-\Delta(\underline{u}_n - \underline{u}_{n+1}) + c(\underline{u}_n - \underline{u}_{n+1}) = F(x, \underline{u}_{n-1}(x)) - F(x, \underline{u}_n(x)) \leqslant 0 \quad \text{in } \Omega,$$
$$\underline{u}_n - \underline{u}_{n+1} = 0 \quad \text{on } \partial\Omega.$$

Thus by comparison, $\underline{u}_n - \underline{u}_{n+1} \leqslant 0$ on $\overline{\Omega}$. This proves (1). The proof of (3) is similar.

When $n = 0$, the inequality (2) is one of our assumptions. Assume inductively that $\overline{u}_{n-1}(x) \geqslant \underline{u}_{n-1}(x)$, then

$$-\Delta(\overline{u}_n - \underline{u}_n) + c(\overline{u}_n - \underline{u}_n) = F(x, \overline{u}_{n-1}(x)) - F(x, \underline{u}_{n-1}(x)) \geqslant 0 \quad \text{in } \Omega,$$
$$\overline{u}_n - \underline{u}_n = 0 \quad \text{on } \partial\Omega.$$

Thus $\overline{u}_n - \underline{u}_n \geqslant 0$ on $\overline{\Omega}$.

We obtained a sequence of continuous functions $\overline{u}_n, \underline{u}_n$ satisfying

$$\underline{u}_{n-1}(x) \leqslant \underline{u}_n(x) \leqslant \overline{u}_n(x) \leqslant \overline{u}_{n-1}(x) \quad \text{on } \overline{\Omega}.$$

By the monotone convergence theorem, both of the limits

$$u_\infty(x) \equiv \lim_{n\to\infty} \underline{u}_n(x), \tag{6.3}$$
$$u^\infty(x) \equiv \lim_{n\to\infty} \overline{u}_n(x) \tag{6.4}$$

exist. We next show that $u_\infty \in C(\overline{\Omega}) \cap C^2(\Omega)$ and satisfies the equation. By Theorem 2.11, for some $\alpha \in (0,1)$,

$$|\underline{u}_n|_{\alpha,\Omega'} \leqslant C.$$

It follows from the Schauder estimates (Theorem 3.2) that, for any $\Omega'' \subset\subset \Omega'$,

$$|\underline{u}_n|_{2,\alpha,\Omega''} \leqslant C,$$

where the constant C is independent of n. Since $C^{2,\alpha}(\overline{\Omega}'') \hookrightarrow C^2(\overline{\Omega}'')$ is compact (Ascoli–Arzelà theorem), a subsequence must converge in $C^2(\overline{\Omega}'')$ and this limit must be u_∞, by (6.3).

We claim that the convergence $\underline{u}_n \to u_\infty$ in $C^2(\overline{\Omega}'')$ is valid for the whole sequence, not just for a subsequence. In fact, if this were not true, then there would exist $\{n_j\}$ such that

$$\|\underline{u}_{n_j} - u_\infty\|_{C^2(\overline{\Omega}'')} \geqslant \epsilon_0 > 0 \quad \text{for } j \geqslant 1.$$

The above argument shows that a further subsequence $\{u_{n_{j_k}}\}$ must converge in $C^2(\overline{\Omega}'')$ and the limit must again be u_∞, by (6.3). This is a contradiction.

We have proved that derivatives up to second order converge on any compact subset of Ω. Therefore $u_\infty \in C^2(\Omega)$, and

$$-\Delta u_\infty + cu_\infty = \lim_{n\to\infty}(-\Delta \underline{u}_n + c\underline{u}_n) = \lim_{n\to\infty} F(x, \underline{u}_{n-1}) = F(x, u_\infty(x)) \text{ in } \Omega.$$

Since $\underline{u}_1(x) \leqslant u_\infty(x) \leqslant \overline{u}_1(x)$, it is also clear that

$$|u_\infty(x) - g(x)| \leqslant \max\Big(|\overline{u}_1(x) - g(x)|, |\underline{u}_1 - g(x)|\Big).$$

The continuity of $\overline{u}_1(x)$ and $\underline{u}_1(x)$ on $\overline{\Omega}$ implies that $u_\infty \in C(\overline{\Omega})$ and

$$u_\infty = g \quad \text{on } \partial\Omega.$$

We can similarly prove that u^∞ is also a solution. For our method of construction we know that $\underline{u} \leqslant u_\infty \leqslant u^\infty \leqslant \overline{u}$. □

Remark 6.1. The procedure used in the proof of this theorem, i.e., using that the compactness plus the uniqueness of the limit implies the convergence of the full sequence (not just a subsequence), is a very useful technique.

Remark 6.2. In general, it is not clear whether u_∞ coincides with u^∞.

Remark 6.3. The upper-lower solution method applies to equations with a comparison principle. It is applicable, for instance, to parabolic equations.

Remark 6.4. The upper-lower solution method applies to systems of two equations with certain structures and carefully defined upper and lower solutions.

In many of our examples $f(x,u)$ and $g(x)$ are nonnegative, so that 0 is a lower solution. In this case the system admits a solution if one can find a upper solution. However, not all systems have upper solutions.

Example 6.1. Consider the Gelfand problem on a bounded smooth domain Ω:

$$-\Delta u = \lambda \exp u \quad \text{in } \Omega, \tag{6.5}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{6.6}$$

Theorem 6.2. *Let $\lambda \geqslant \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$, i.e.,*

$$\begin{aligned} -\Delta\phi = \lambda_1\phi, \quad \phi > 0 \quad &\text{in } \Omega,\\ \phi = 0 \quad &\text{on } \partial\Omega,\\ \int_\Omega \phi(x)dx = 1.& \end{aligned}$$

Then (6.5) and (6.6) has no solutions.

Proof. We assume that a solution u exists. Then u is positive in Ω by the maximum principle (Theorem 2.10), and

$$\begin{aligned} 0 = \int_\Omega (u\Delta\phi - \phi\Delta u)dx &= \int_\Omega (-\lambda_1 u\phi + \phi\lambda \exp u)dx \\ &> \int_\Omega \{-\lambda_1 u\phi + \phi\lambda(u+1)\}. \end{aligned}$$

The right-hand side of the above inequality is positive, since $\lambda \geqslant \lambda_1$, which is a contradiction. □

In this case it is natural to expect that the corresponding parabolic problem cannot have a global solution. We leave it as an exercise.

Remark 6.5. The solution of (6.5) and (6.6) exists if $\lambda > 0$ is small. Its proof is left as an exercise.

Remark 6.6. The system (6.5) and (6.6) may indeed have more than one positive solutions. Take the special case that $n = 1$, $u(-x) = u(x)$, $\Omega = (-1,1)$. The system (6.5) and (6.6) is reduced to an ODE:

$$\begin{aligned} u'' + \lambda \exp u = 0, \quad 0 < x < 1,\\ u'(0) = 0, \quad u(1) = 0. \end{aligned}$$

Multiplying the equation with u' and integrating over $[0,x]$, we obtain

$$\frac{1}{2}u'^2(x) + \lambda \exp(u(x)) = \lambda \exp \alpha, \quad \alpha = u(0) > 0,$$

from which we can solve u explicitly (noticing that $u' < 0$ for $x > 0$),

$$u(x) = \alpha - 2 \log \cosh \left(\frac{1}{2}x\sqrt{2\lambda \exp \alpha}\right).$$

Thus $u(1) = 0$, and $u(x) > 0$ for $0 \leqslant x < 1$ if and only if

$$\exp \alpha = \cosh^2 \left(\frac{1}{2}\sqrt{2\lambda \exp \alpha}\right),$$

or, equivalently,

$$\lambda = \frac{1}{2}\exp(-\alpha) \cdot \log^2 \left(\frac{1 + \sqrt{1 - \exp(-\alpha)}}{1 - \sqrt{1 - \exp(-\alpha)}}\right).$$

From this equality we see that there exists a critical value λ^* such that:

(i) There are two solutions for $\lambda \in (0, \lambda^*)$ for some $\lambda^* \approx 0.45$,
(ii) The two solutions merge into one at $\lambda = \lambda^*$, and
(iii) There are no solutions when $\lambda > \lambda^*$.

6.2 The Moving Plane Method (Gidas–Ni–Nirenberg)

If in (6.5) and (6.6) the domain Ω is a ball of radius R, i.e., $\Omega = \{x \in \mathbb{R}^n; \ |x| < R\}$, are there non-radial solutions? The answer is no. In fact, the symmetry properties were studied in Gidas–Ni–Nirenberg [60] for a large class of such problems using the very powerful moving plane method.

We begin with a Hopf Lemma for *nonnegative* solutions.

Lemma 6.3. *Let $\partial\Omega$ satisfies the interior sphere condition at $x_0 \in \partial\Omega$. Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

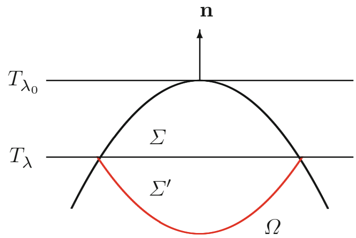
$$-a^{ij}D_{ij}u + b^iD_iu + cu \geqslant 0 \quad \text{in } \Omega,$$

where a^{ij}, b^i, c are continuous functions on $\overline{\Omega}$ and a^{ij} satisfies the uniform ellipticity condition. If $u \geqslant 0$, $u \not\equiv 0$ and $u(x_0) = 0$, then

$$\limsup_{\sigma \to 0} \frac{u(x_0 + \sigma\eta) - u(x_0)}{\sigma} < 0 \tag{6.7}$$

for any direction η such that $\eta \cdot \mathbf{n} > 0$, where $\mathbf{n}$ is the exterior normal vector.

Fig. 6.1 The moving plane



Proof. Writing $c = c^+ - c^-$, we then have

$$-a^{ij}D_{ij}u + b^iD_iu + c^+u \geqslant c^-u \geqslant 0, \tag{6.8}$$

from which the lemma follows by Theorem 2.9. □

The moving plane method. We assume that $\partial\Omega \in C^2$. Take a unit vector $\mathbf{n}$ and define a family of "moving planes" parameterized by $\lambda \in \mathbb{R}$, i.e., $T_\lambda = \{x \in \mathbb{R}^n;\ \mathbf{n} \cdot x = \lambda\}$. We assume that there exists a λ_0 such that $T_{\lambda_0} \cap \overline{\Omega} \neq \emptyset$ and $T_\lambda \cap \overline{\Omega} = \emptyset$ for $\lambda > \lambda_0$ (see Fig. 6.1).

For any $x \in \mathbb{R}^n$, we denote by x^λ its reflection point through T_λ. We can then define the *open cap* $\Sigma(\lambda)$ and its reflection through T_λ, as follows.

$$\Sigma(\lambda) = \{x \in \Omega;\ \ \mathbf{n} \cdot x > \lambda\},$$
$$\Sigma'(\lambda) = \{x^\lambda;\ \ x \in \Sigma(\lambda)\}.$$

It is clear that $\Sigma(\lambda) = \emptyset$ if $\lambda \geqslant \lambda_0$ and $\Sigma(\lambda) \neq \emptyset$ if $\lambda < \lambda_0$.

For $0 < \lambda_0 - \lambda \ll 1$, it is clear that $\Sigma'(\lambda) \subset \Omega$. We can decrease λ (moving the planes) until one of the following happens (*shape considerations*):

(i) $\Sigma'(\lambda)$ becomes internally tangent to $\partial\Omega$ at some point $p \notin T_\lambda$ (we have $p \in (\partial\Sigma'(\lambda) \setminus T_\lambda) \cap \partial\Omega$ in this case), or
(ii) T_λ is orthogonal to $\partial\Omega$ at some point $q \in T_\lambda \cap \partial\Omega$.

Define

$$\lambda_1 = \sup\{\lambda < \lambda_0;\quad \text{(i) or (ii) occurs}\ \}. \tag{6.9}$$

Then neither (i) nor (ii) will occur when $\lambda > \lambda_1$. The cap $\Sigma(\lambda_1)$ is called the *maximal cap* associated with this moving plane process. Note that $\Sigma'(\lambda_1) \subset \Omega$.

Lemma 6.4. *Let $\partial\Omega \in C^2$, $x_0 \in \partial\Omega$ and that $\mathbf{n}$ is the exterior normal vector on $\partial\Omega$. If for some small $\varepsilon > 0$, $u \in C^2(\overline{\Omega \cap B_\varepsilon(x_0)})$ satisfies*

$$u(x) = 0 \quad \text{on } \partial\Omega \cap B_\varepsilon(x_0), \tag{6.10}$$
$$\frac{\partial u}{\partial n}(x_0) = 0, \tag{6.11}$$

then

$$\nabla u(x_0) = 0, \tag{6.12}$$

$$\frac{\partial^2 u}{\partial n^2}(x_0) = \Delta u(x_0), \qquad \frac{\partial^2 u}{\partial \gamma^2}(x_0) = \Delta u(x_0)(\mathbf{n}(x_0)\cdot\gamma)^2. \tag{6.13}$$

Proof. Equation (6.12) is trivial. To prove (6.13), we assume without loss of generality that the axis x_n is in the direction of $\mathbf{n}(x_0)$. Writing

$$\partial\Omega \cap B_\varepsilon(x_0) : x_n = f(x_1, \cdots, x_{n-1}),$$

where $f \in C^2$ and satisfies

$$f_{x_j}(x_1^0, \cdots, x_{n-1}^0) = 0, \qquad x_0 = (x_1^0, \cdots, x_{n-1}^0, x_n^0). \tag{6.14}$$

Differentiating the relation $u(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1})) \equiv 0$ twice in x_j, using also (6.14) and the assumption that $u_{x_n}(x_0) = 0$, we obtain

$$u_{x_i x_j}(x_0) = 0 \quad \text{for } i, j = 1, 2, \cdots, n-1,$$

from which the first equality of (6.13) follows. Writing $\frac{\partial^2 u}{\partial \gamma^2}$ in terms of $u_{x_i x_j}$ ($1 \leqslant i$, $j \leqslant n$), we obtain the second equality of (6.13). □

Lemma 6.5. *Let $\partial\Omega \in C^2$, $x_0 \in \partial\Omega$ and $\mathbf{n}(x_0)\cdot\gamma > 0$, where $\mathbf{n}$ is the exterior normal vector on $\partial\Omega$. Take $\varepsilon > 0$ small enough so that $\mathbf{n}(x)\cdot\gamma > 0$ for all $x \in \partial\Omega \cap B_\varepsilon(x_0)$. If $u \in C^2(\overline{\Omega \cap B_\varepsilon(x_0)})$ and $f \in C^1(\mathbb{R})$ satisfy*

$$\Delta u + f(u) = 0 \quad \text{in } \Omega \cap B_\varepsilon(x_0), \tag{6.15}$$

$$u(x) > 0 \quad \text{in } \Omega \cap B_\varepsilon(x_0), \tag{6.16}$$

$$u(x) = 0 \quad \text{on } \partial\Omega \cap B_\varepsilon(x_0), \tag{6.17}$$

then there exists a $\delta \in (0, \varepsilon)$ such that $\frac{\partial u}{\partial \gamma} < 0$ on $\Omega \cap B_\delta(x_0)$.

Proof. By continuity, $\mathbf{n}(x)\cdot\gamma > 0$ for $x \in \partial\Omega \cap B_\varepsilon(x_0)$ if $\varepsilon \ll 1$. Since $u = 0$ on $\partial\Omega \cap B_\varepsilon(x_0)$ is the minimum value, we must have $\frac{\partial u}{\partial \gamma}(x) \leqslant 0$ for $x \in \partial\Omega \cap B_\varepsilon(x_0)$.

If $f(0) \geqslant 0$, then for $c(x) = \displaystyle\int_0^1 f'(\tau u(x))d\tau$,

$$-\Delta u - c(x)u = f(0) \geqslant 0 \quad \text{in } \Omega \cap B_\varepsilon(x_0),$$

so that, by Lemma 6.3, $\frac{\partial u}{\partial \gamma}(x_0) < 0$. By continuity, the conclusion holds if $\delta > 0$ is small enough.

We next consider the case $f(0) < 0$. If the conclusion is not true, then we must have $\frac{\partial u}{\partial \gamma}(x_0) = 0$ (otherwise we can use the continuity argument). It follows that $\frac{\partial u}{\partial n}(x_0) = 0$, and by Lemma 6.4, we have $\frac{\partial^2 u}{\partial \gamma^2}(x_0) = \Delta u(x_0)(\mathbf{n}(x_0)\cdot\gamma)^2 = -f(0)(\mathbf{n}(x_0)\cdot\gamma)^2 > 0$. By continuity $\frac{\partial^2 u}{\partial \gamma^2}(x) > 0$ on $\overline{\Omega \cap B_\delta(x_0)}$ if $\delta > 0$ is small.

Since $\frac{\partial u}{\partial \gamma}(x) \leqslant 0$ on $\partial\Omega \cap B_\varepsilon(x_0)$, we must have $\frac{\partial u}{\partial \gamma}(x) < 0$ on $\Omega \cap B_\delta(x_0)$ by the monotonicity of $\frac{\partial u}{\partial \gamma}(x)$ in γ direction. □

Consider now the problem

$$-\Delta u = f(u) \quad \text{in } \Omega, \tag{6.18}$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{6.19}$$
$$u \geqslant 0 \quad \text{in } \Omega, \tag{6.20}$$

where we assume that Ω is a bounded domain with $\partial\Omega \in C^2$ and $f \in C^1(\mathbb{R})$.

We now consider the moving plane procedure:

Start moving the plane. By Lemma 6.5, for $0 < \lambda_0 - \lambda \ll 1$, $\gamma = \mathbf{n}(x_0)$, we have

$$\frac{\partial u}{\partial \gamma}(x) < 0 \quad \text{for } x \in \Sigma(\lambda) \cup T_\lambda \cup \Sigma'(\lambda), \tag{6.21}$$
$$u(x) < u(x^\lambda) \quad \text{for } x \in \Sigma(\lambda). \tag{6.22}$$

Continue moving. We want to move the plane T_λ from $0 < \lambda_0 - \lambda \ll 1$ to a smaller λ while preserving the above monotonicity property. We divide the proof into several lemmas.

Lemma 6.6. *Let $\gamma = \mathbf{n}(x_0)$. If $\lambda \in [\lambda_1, \lambda_0)$ satisfies*

$$\frac{\partial u}{\partial \gamma}(x) \leqslant 0 \quad \textit{for } x \in \Sigma(\lambda), \tag{6.23}$$
$$u(x) \leqslant u(x^\lambda) \quad \textit{for } x \in \Sigma(\lambda), \tag{6.24}$$
$$u(x) \not\equiv u(x^\lambda) \quad \textit{for } x \in \Sigma(\lambda), \tag{6.25}$$

then

$$u(x) < u(x^\lambda) \quad \textit{for } x \in \Sigma(\lambda), \tag{6.26}$$
$$\frac{\partial u}{\partial \gamma}(x) < 0 \quad \textit{for } x \in \Omega \cap T_\lambda. \tag{6.27}$$

Proof. Assume without loss of generality $\gamma = \mathbf{n}(x_0) = (1, 0, \cdots, 0)$ (see Fig. 6.1). Then for $x \in \Sigma(\lambda)$, we have $x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n)$.

Consider the function $\phi(x) := u(x^\lambda) - u(x)$ for $x \in \Sigma(\lambda)$. Clearly,

$$\begin{aligned} -\Delta\phi(x) &= f(u(x^\lambda)) - f(u(x)) = c(x)\phi(x) \quad \text{in } \Sigma(\lambda), \\ \phi(x) &\geqslant 0 \quad \text{in } \Sigma(\lambda), \\ \phi(x) &= 0 \quad \text{on } \Omega \cap T_\lambda. \end{aligned}$$

By the maximum principle and Lemma 6.3, $\phi(x) > 0$ for $x \in \Sigma(\lambda)$ and $\frac{\partial \phi}{\partial \gamma}(x) > 0$ for $x \in \Omega \cap T_\lambda$. The lemma follows immediately. □

The next lemma will enable us to move our plane all the way to $\lambda = \lambda_1$ (the value λ_1 corresponds to the maximal cap). This is the *key step* for the moving plane method.

Lemma 6.7. *Set* $\gamma = \mathbf{n}(x_0) = (1, 0, \cdots, 0)$. *For* $\lambda \in (\lambda_1, \lambda_0)$, *we have*

$$u(x) < u(x^\lambda) \quad \textit{for } x \in \Sigma(\lambda), \tag{6.28}$$

$$\frac{\partial u}{\partial \gamma}(x) < 0 \quad \textit{for } x \in \Sigma(\lambda). \tag{6.29}$$

Proof. By the procedures in (6.21) and (6.22), we have for $0 < \lambda_0 - \lambda \ll 1$,

$$\frac{\partial u}{\partial x_1}(x) < 0 \qquad \text{and} \quad u(x) < u(x^\lambda) \quad \text{for } x \in \Sigma(\lambda). \tag{6.30}$$

Decrease λ until it cannot be further decreased, i.e., decrease λ until it reaches

$$\lambda^* = \inf\{\overline{\lambda} \in [\lambda_1, \lambda_0); \;\; (6.30) \text{ holds for all } \lambda \in (\overline{\lambda}, \lambda_0)\}.$$

By the definition of λ^*, (6.30) is satisfied for $\lambda \in (\lambda^*, \lambda_0)$. By continuity,

$$\frac{\partial u}{\partial x_1}(x) < 0 \qquad \text{and} \quad u(x) \leqslant u(x^\lambda) \quad \text{for } x \in \Sigma(\lambda) \quad \text{for } \lambda = \lambda^*. \tag{6.31}$$

We claim that $\lambda^* = \lambda_1$. Assume the contrary. Then $\lambda^* > \lambda_1$, and the assumptions of Lemma 6.6 are satisfied. It follows that

$$u(x) < u(x^\lambda) \quad \text{for } x \in \Sigma(\lambda), \quad \lambda = \lambda^*, \tag{6.32}$$

$$\frac{\partial u}{\partial \gamma}(x) < 0 \qquad \text{for } x \in \Omega \cap T_\lambda, \quad \lambda = \lambda^*. \tag{6.33}$$

We can apply Lemma 6.5 at the point $x \in \partial\Omega \cap T_{\lambda^*}$ (notice that since $\lambda^* \neq \lambda_1$, γ is not tangent to $\partial\Omega$, by the definition of λ_1) combined with (6.33) to conclude

$$\frac{\partial u}{\partial \gamma}(x) < 0 \quad \text{for } x \in \Sigma(\lambda), \quad \lambda = \lambda^* - \varepsilon, \tag{6.34}$$

for all small $\varepsilon > 0$. By definition of λ^*, there exist $\lambda_j \nearrow \lambda^*$ and $x_j \in \Sigma(\lambda_j)$ such that

$$u(x_j) \geqslant u(x_j^{\lambda_j}). \tag{6.35}$$

Taking a subsequence if necessary, $x_j \to x \in \overline{\Sigma(\lambda^*)}$ and therefore $x_j^{\lambda_j} \to x^{\lambda^*} \in \overline{\Sigma'(\lambda^*)}$.

By (6.32), we have $x \in \partial\Sigma(\lambda^*)$. If $x \notin T_{\lambda^*}$, then we get a contradiction since $u = 0$ on $\partial\Omega$ and $u > 0$ in Ω. Thus $x \in T_{\lambda^*}$. It follows that $|x_j - x_j^{\lambda_j}| \to 0$, and (6.35) contradicts (6.33). $\square$

Corollary 6.8. *If $\frac{\partial u}{\partial \gamma}(x) = 0$ for some $x \in \Omega \cap T_{\lambda_1}$, then $u(x) \equiv u(x^{\lambda_1})$, and*

$$\Omega = \Sigma(\lambda_1) \cup \Sigma'(\lambda_1) \cup [T_{\lambda_1} \cap \Omega]. \tag{6.36}$$

Proof. By Lemmas 6.6 and 6.7, we have $u(x) \equiv u(x^{\lambda_1})$. Since $u > 0$ in Ω and $u = 0$ on $\partial\Omega$, we conclude (6.36). $\square$

Theorem 6.9. *Consider the system (6.18)–(6.20) with $\Omega = B_R$. If $u \in C^2(\overline{\Omega})$ is a positive solution with $f \in C^1(\mathbb{R})$, then $u(x) = u(r)$ where $r = |x|$ and $u'(r) < 0$ for $r \in (0, R)$.*

Proof. We can use the moving plane method in any direction. Take γ to be in the x_1 direction. It is clear that T_{λ_1} is a plane through the origin. Thus $u_{x_1} < 0$ for $x_1 > 0$. Taking γ to be in the $-x_1$ direction we obtain $u_{x_1} > 0$ for $x_1 < 0$. Thus $u_{x_1} = 0$ for $x_1 = 0$ and therefore by Corollary 6.8 u is symmetric in x_1. Since γ is an arbitrary direction, we conclude the theorem. $\square$

6.3 The Moving Plane Method on Unbounded Domains

One of the difficulties for extending the moving plane method to unbounded domains is finding a suitable place to start the process. Consider the problem of finding nonnegative solutions to the equation

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^n. \tag{6.37}$$

Theorem 6.10. *(i) If $1 < p < \frac{n+2}{n-2}$ for $n \geqslant 3$ and $1 < p < \infty$ if $n = 1, 2$, then the only nonnegative solution to (6.37) is the trivial solution $u \equiv 0$.*

(ii) If $p = \frac{n+2}{n-2}$, then the nonnegative solutions to (6.37) are radially symmetric, and must be of the form

$$u(x) = c_1 \Big(c_2 + |x - x_0|^2\Big)^{-(n-2)/2},$$

for some $x_0 \in \mathbb{R}^n$ and either $c_1 = 0$, or $c_2 > 0$, and $c_1^{p-1} = n(n-2)c_2$.

Remark 6.7. The classification of the solutions in the case $p = \frac{n+2}{n-2}$ is related to the study of the Yamabe problem. This result was proved by Gidas–Ni–Nirenberg [61] under the additional assumption that $u(x) = O(|x|^{-(n-2)})$ for $|x| \gg 1$. This additional assumption was removed by Caffarelli–Gidas–Spruck [16]. The proof here is a simplified version by Chen–Li [18]. Related results can also be found in Gidas–Spruck [62, 63].

Proof. In the case $n = 1$, we have

$$u''(x) \leqslant 0, \qquad u(x) \geqslant 0, \quad -\infty < x < \infty.$$

This implies that $u'(x) \equiv 0$. (If $u'(x_0) < 0$ for some x_0, then $u'(x) \leqslant u'(x_0) < 0$ for $x > x_0$ and u cannot remain positive for $x \gg 1$; similarly, u cannot remain positive for $x \ll -1$ if $u'(x_0) > 0$ for some x_0). Thus $u \equiv \text{const}. = 0$.

In the case $n = 2$, we have

$$-\Delta u(x) \geqslant 0, \qquad u(x) \geqslant 0, \quad x \in \mathbb{R}^2.$$

A super-harmonic function on $\mathbb{R}^2$ which is bounded from below must be constant (left as an exercise). Thus $u \equiv \text{const}. = 0$.

We now consider the case $n \geqslant 3$, $1 < p < \frac{n+2}{n-2}$, we shall prove that u is symmetric with respect to any point in $\mathbb{R}^n$, and hence u must be constant, and furthermore from the equation this constant must be 0.

We take an arbitrary point in $\mathbb{R}^n$. Without loss of generality, we assume that this point is the origin.

Step 1. Kelvin's transform. For each $\mu > 0$, it is clear that the function $v(x) = \mu^{2/(p-1)} u(\mu x)$ satisfies the same equation in $\mathbb{R}^n$, and

$$\sup_{x \in B_1(0)} v(x) \leq \mu^{2/(p-1)} \sup_{x \in B_\mu(0)} u(x) < \left(\frac{n-2}{2\sqrt{p}}\right)^{2/(p-1)}, \tag{6.38}$$

provided we take μ to be small enough. We fix such a μ.

Suppose that $u \not\equiv 0$. Then by the maximum principle (Theorem 2.10), $\varepsilon = \min\limits_{\overline{B}_1(0)} v(x) > 0$. Notice that

$$\lim_{M \to \infty} \inf_{|x| > M} \left(v(x) - \frac{\varepsilon}{|x|^{n-2}} \right) \geq 0.$$

Thus a simple application of the comparison principle implies that

$$v(x) \geq \frac{\varepsilon}{|x|^{n-2}} - \frac{\varepsilon}{M^{n-2}} \qquad \text{for } 1 < |x| < M.$$

By letting $M \to \infty$, we derive

$$v(x) \geq \frac{\varepsilon}{|x|^{n-2}} \qquad \text{for } |x| \geq 1. \tag{6.39}$$

We now introduce the Kelvin's inversion $w(x) = \dfrac{1}{|x|^{n-2}} v\left(\dfrac{x}{|x|^2}\right)$. Then w satisfies the equations:

$$-\Delta w = \frac{1}{|x|^\alpha} w^p \quad \text{in } \mathbb{R}^n, \tag{6.40}$$

$$w(x) \geq \varepsilon \quad \text{on } B_1(0), \tag{6.41}$$

$$w(x) < \Big(\frac{n-2}{2\sqrt{p}}\Big)^{2/(p-1)} \frac{1}{|x|^{n-2}} \quad \text{for } |x| \geq 1, \tag{6.42}$$

where $\alpha = (n+2) - (n-2)p$. The function w may have a singularity at $x = 0$.

Introduce the notation for the moving planes

$$\begin{aligned} &\Sigma_\lambda = \{(x_1, \cdots, x_n);\ x_n < \lambda\}, \quad \lambda \in R^1, \\ &T_\lambda = \{(x_1, \cdots, x_n);\ x_n = \lambda\}, \\ &\widetilde{\Sigma}_\lambda = \overline{\Sigma}_\lambda \setminus \{(0, \cdots, 0, 2\lambda)\}. \end{aligned}$$

Set

$$\begin{aligned} &\psi_\lambda(x) = w_\lambda(x) - w(x), \quad x \in \widetilde{\Sigma}_\lambda, \\ &w_\lambda(x) = w(x_1, \cdots, x_{n-1}, 2\lambda - x_n) = w(x^\lambda). \end{aligned}$$

Clearly, for $\lambda < 0$, $|x^\lambda| \leqslant |x|$, and

$$\begin{aligned} -\Delta \psi_\lambda &= \frac{1}{|x|^\alpha}(w_\lambda^p - w^p) + \Big(\frac{1}{|x^\lambda|^\alpha} - \frac{1}{|x|^\alpha}\Big) w_\lambda^p \\ &\geqslant \frac{1}{|x|^\alpha}(w_\lambda^p - w^p) \quad \text{in } \widetilde{\Sigma}_\lambda. \end{aligned} \tag{6.43}$$

Step 2. Start the moving plane process. Let us prove:

$$\psi_\lambda \geq 0 \quad \text{in } \widetilde{\Sigma}_\lambda \quad \text{if } -\lambda \gg 1. \tag{6.44}$$

Define

$$\phi_\lambda(x) = |x|^\beta \psi_\lambda(x), \quad \beta = \frac{n-2}{2}.$$

Then

$$-\Delta \phi_\lambda + \frac{2\beta}{|x|^2}\, x \cdot \nabla \phi_\lambda + \frac{\beta^2}{|x|^2} \phi_\lambda \geqslant \frac{|x|^\beta}{|x|^\alpha}(w_\lambda^p - w^p) \quad \text{in } \widetilde{\Sigma}_\lambda.$$

We rewrite this inequality as

$$-\Delta \phi_\lambda + \frac{2\beta}{|x|^2}\, x \cdot \nabla \phi_\lambda + c(x)\phi_\lambda \geqslant 0 \quad \text{in } \widetilde{\Sigma}_\lambda, \tag{6.45}$$

where

$$c(x) = \frac{\beta^2}{|x|^2} - \frac{p\xi^{p-1}(x)}{|x|^\alpha}, \qquad \xi(x) \text{ is between } w_\lambda(x) \text{ and } w(x).$$

We claim that

$$c(\widetilde{x}) > 0 \qquad \text{if } \widetilde{x} \in \widetilde{\Sigma}_\lambda, \quad |\widetilde{x}| > 1 \text{ and } \phi_\lambda(\widetilde{x}) \leq 0. \tag{6.46}$$

In fact, if $\phi_\lambda(\widetilde{x}) \leq 0$ and $|\widetilde{x}| > 1$, then by (6.42),

$$0 \leq w_\lambda(\widetilde{x}) \leq \xi(\widetilde{x}) \leq w(\widetilde{x}) \leq \Big(\frac{n-2}{2\sqrt{p}}\Big)^{2/(p-1)} \frac{1}{|\widetilde{x}|^{n-2}}.$$

and thus

$$c(\widetilde{x}) > \frac{\beta^2}{|\widetilde{x}|^2} - \Big(\frac{n-2}{2\sqrt{p}}\Big)^2 \frac{p}{|\widetilde{x}|^{\alpha+(n-2)(p-1)}} > 0.$$

Near the point $(0,\cdots,0,2\lambda)$ (which is a possible singularity point of $\phi_\lambda(x)$), we have, by (6.41) and (6.42),

$$\begin{aligned}
\phi_\lambda(x) &\geq |x|^\beta\Big(\varepsilon - w(x)\Big) \\
&\geq |x|^\beta\Big(\varepsilon - \frac{C}{|x|^{n-2}}\Big) \qquad \Big(C \text{ is independent of } \lambda\Big) \\
&\geq |x|^\beta\Big(\varepsilon - \frac{C}{|\lambda|^{n-2}}\Big) > 0 \quad \text{for } x \in \widetilde{\Sigma}_\lambda,\ |x-(0,\cdots,0,2\lambda)| \leq 1,
\end{aligned} \tag{6.47}$$

if $-\lambda \gg 1$. It is also clear that

$$\begin{aligned}
\phi_\lambda(x) &= 0 \quad \text{on } T_\lambda, \\
|\phi_\lambda(x)| &\leqslant \frac{C|x|^\beta}{|x|^{n-2}} \to 0 \quad \text{as } x \to \infty.
\end{aligned}$$

Thus if ϕ_λ is negative somewhere inside $\widetilde{\Sigma}_\lambda$, the negative minimum of ϕ_λ on $\widetilde{\Sigma}_\lambda$ must be reached at a finite interior point x^* of $\widetilde{\Sigma}_\lambda \setminus \{|x-(0,\cdots,0,2\lambda)| \leq 1\}$. We clearly have $|x^*| > 1$ if $-\lambda > 1$. Thus $c(x^*) > 0$ by (6.47) and we get a contradiction to (6.45). Thus (6.44) is established. We remark here that this process is also valid when $p = (n+2)/(n-2)$.

Step 3. Move the plane. Let λ_0 be the supremum of all negative $\lambda' s$ such that $\psi_\lambda \geq 0$ in $\widetilde{\Sigma}_\lambda$, namely,

$$\lambda_0 = \sup\{\lambda < 0;\ \psi_\mu(x) \geq 0 \text{ in } \widetilde{\Sigma}_\mu \text{ for all } -\infty < \mu < \lambda\}.$$

We claim that

$$\lambda_0 = 0 \quad \text{if } p < \frac{n+2}{n-2}. \tag{6.48}$$

By Step 2, this λ_0 exists and is a finite number. Suppose on the contrary that $\lambda_0 < 0$. By the continuity, $\phi_{\lambda_0} = |x|^\beta \psi_{\lambda_0} \geq 0$ in $\widetilde{\Sigma}_{\lambda_0}$. Since $\alpha = n+2-(n-2)p > 0$

and $\lambda_0 < 0$, $\psi_{\lambda_0} \not\equiv 0$ in $\widetilde{\Sigma}_{\lambda_0}$ by (6.43). Therefore by the maximum principle (Theorem 2.10)

$$\psi_{\lambda_0} > 0 \quad \text{in } \overline{\Sigma}_{\lambda_0} \setminus \Big(T_{\lambda_0} \cup \{(0, \cdots, 0, 2\lambda_0)\}\Big). \tag{6.49}$$

In particular,

$$\delta = \inf_{\{|x-(0,\cdots,0,2\lambda_0)|=|\lambda_0|/2\}} \psi_{\lambda_0}(x) > 0. \tag{6.50}$$

There may be a singularity at the point $(0, \cdots, 0, 2\lambda_0)$ for ψ_{λ_0} (and hence for ϕ_{λ_0}), so we construct the auxiliary function $h_\epsilon(x)$:

$$\begin{aligned} \Delta h_\epsilon(x) = 0 \quad & \text{for } \epsilon < |x - (0, \cdots, 0, 2\lambda_0)| < \frac{1}{2}|\lambda_0|, \\ h_\epsilon(x) = \delta \quad & \text{for } |x - (0, \cdots, 0, 2\lambda_0)| = \frac{1}{2}|\lambda_0|, \\ h_\epsilon(x) = 0 \quad & \text{for } |x - (0, \cdots, 0, 2\lambda_0)| = \epsilon. \end{aligned}$$

The maximum principle implies that,

$$\psi_{\lambda_0}(x) \geq h_\epsilon(x) \quad \text{for } \epsilon \leq |x - (0, \cdots, 0, 2\lambda_0)| \leq \frac{1}{2}|\lambda_0|.$$

Letting $\epsilon \to 0+$, and noticing also that $\lim\limits_{\epsilon \to 0+} h_\epsilon(x) \equiv \delta$ (for $n \geqslant 2$, a bounded harmonic function on a punctured disk has a removable singularity), we get

$$\psi_{\lambda_0}(x) \geq \delta \quad \text{for } 0 < |x - (0, \cdots, 0, 2\lambda_0)| \leq \frac{1}{2}|\lambda_0|. \tag{6.51}$$

Since $\phi_{\lambda_0}(x) \geq |\lambda_0|^\beta \psi_{\lambda_0}(x)$ for all $x \in \widetilde{\Sigma}_{\lambda_0}$,

$$\lim_{\lambda \to \lambda_0} \inf_{\{|x-(0,\cdots,0,2\lambda_0)| \leq |\lambda_0|/2\}} \phi_\lambda(x) \geq \inf_{\{|x-(0,\cdots,0,2\lambda_0)| \leq |\lambda_0|/2\}} \phi_{\lambda_0}(x) \geq \delta |\lambda_0|^\beta. \tag{6.52}$$

Since λ_0 is the supremum, there exist $\lambda_k \searrow \lambda_0$ such that

$$\inf_{x \in \widetilde{\Sigma}_{\lambda_k}} \phi_{\lambda_k}(x) < 0. \tag{6.53}$$

Clearly, $\lim\limits_{|x| \to +\infty} \phi_{\lambda_k}(x) = 0$. Since $\phi_\lambda(x) = 0$ on T_λ, the infimum in (6.53) cannot be reached on T_{λ_k}. Recalling also (6.52), the infimum in (6.53) must be achieved at some finite interior point $x^k \in \Sigma_{\lambda_k} \setminus B_{|\lambda_0|/2}\Big((0, \cdots, 0, 2\lambda_0)\Big)$ provided $|\lambda_k - \lambda_0|$ is small enough.

If $|x^k| > 1$, then we again obtain a contradiction to (6.45) and (6.46). Thus we must have $|x^k| \leqslant 1$. By choosing a subsequence if necessary, we assume that

$x^k \to x^\infty \in \overline{\Sigma_{\lambda_0} \setminus B_{|\lambda_0|/2}\Big((0,\cdots,0,2\lambda_0)\Big)}$, and $\psi_{\lambda_0}(x^\infty) = 0$. By (6.51), we have $x^\infty \in T_{\lambda_0}$.

Connecting x^k to the plane T_{λ_k} by a line segment in the normal direction (x_n direction) of the plane, we can find a point $\widetilde{x}^k$ on this line segment such that

$$\frac{\partial \phi_{\lambda_k}}{\partial x_n}(\widetilde{x}^k) \geqslant 0,$$

by the mean value theorem. It is clear that $\widetilde{x}^k \to x^\infty$. Taking the limit, we find that

$$\frac{\partial \phi_{\lambda_0}}{\partial x_n}(x^\infty) \geqslant 0,$$

which is a contradiction to Hopf's lemma. We proved that $\lambda_0 = 0$.

Step 4. Finishing the proof for the case $1 < p < (n+2)/(n-2)$. Since $\lambda_0 = 0$, we obtain $w(x_1, \cdots, x_{n-1}, -x_n) \geqslant w(x)$. Using the moving plane method in the oppositive direction we also have $w(x_1, \cdots, x_{n-1}, -x_n) \leqslant w(x)$. In particular, this implies that $u(x)$ is symmetric with respect to the plane $\{x_n = 0\}$. Repeating this argument in arbitrary direction for arbitrary point as the origin, we find that $u(x)$ is symmetric with respect to any plane, therefore it must be a constant. This constant must be zero, by the equation.

Step 5. The case $p = (n+2)/(n-2)$**.** Note that in this case $\alpha = 0$. We can follow Steps 1 and 2 above in this case. However, in Step 3 we cannot conclude $\phi_{\lambda_0} \not\equiv 0$ since we have $\alpha = 0$ in (6.43). We have two possible scenarios: (1) either we can move the plane to the origin $\lambda_0 = 0$ in any possible directions, in which case we must have $u(x) = u(|x|)$; or, (2) in at least one direction, $\phi_{\lambda_0} \equiv 0$ for some $\lambda_0 < 0$. In the latter case we have symmetry of $w(x)$ with respect to the plane $\{x_n = \lambda_0\}$. Then

$$v\Big(\frac{x}{|x|^2}\Big) = \frac{|x|^{n-2}}{|x^{\lambda_0}|^{n-2}} v\Big(\frac{x^{\lambda_0}}{|x^{\lambda_0}|^2}\Big) \quad \text{for } x_n < \lambda_0 < 0.$$

Letting $y = \dfrac{x^{\lambda_0}}{|x^{\lambda_0}|^2}$, we then have

$$v(y) \leqslant \frac{C}{|y|^{n-2}} \quad \text{for } |y| \gg 1. \tag{6.54}$$

Thus v achieves its maximum at a finite point x_0. By rescaling and translation if necessary, we assume that v attains its maximum at 0 and satisfies (6.40)–(6.42). Now we can now apply the moving plane method directly to v and the fact that $v(0) = \max_{\mathbb{R}^n} v(x)$ enables us to move the plane to the origin. It follows that $v(x) = v(|x|)$.

Denote x_0 the maximum of u and $r = |x - x^0|$. Then the problem is reduced to solving the ODE:

$$u''(r) + \frac{n-1}{r}u'(r) + u^{(n+2)/(n-2)}(r) = 0, \quad u'(r) < 0 \quad \text{for } r > 0,$$
$$u'(0) = 0, \quad u(0) > 0.$$

Take c_2 so that $u(0)$ agrees with the solution in the theorem. Then the conclusion of the theorem follows by uniqueness. □

Remark 6.8. The similar results extend to problems with nonlinear boundary conditions (see Exercise 6.4).

Remark 6.9. One important feature for the moving plane method is the simplicity. The main tool used is the maximum principle. The moving plane method is nonetheless an very handy tool for studying symmetry properties of solutions and has been employed by many authors.

6.4 Exercises

6.1. Prove that (6.5) and (6.6) has a solution if $\lambda > 0$ is sufficiently small.

6.2. Consider the problem:

$$u_t - \Delta u = \lambda \exp u \quad \text{in } \Omega \times \{t > 0\},$$
$$u = 0 \quad \text{on } \partial\Omega \times \{t > 0\},$$
$$u(x, 0) = u_0(x) \geqslant 0,$$

where Ω is smooth, and u_0 is smooth and satisfies the compatibility conditions.

1. If $\lambda \geqslant \lambda_1$, where λ_1 is the first eigenvalue for the $-\Delta$ with zero Dirichlet boundary condition. Prove that all solutions blow up in finite time.
2. Prove that if $\lambda > 0$ is sufficiently small, then there exist solutions global in time.

6.3. Prove that a super-harmonic defined on $\mathbb{R}^2$ which is bounded from below must be constant. This result is not true in $\mathbb{R}^3$, as shown in Theorem 6.10(ii).

6.4.* (See [23, 72, 96]) For the equation on the half-space with nonlinear boundary condition:

$$-\Delta u = 0, \quad u(x) \geqslant 0, \quad x \in \mathbb{R}^n_+,$$
$$\frac{\partial u}{\partial n} = u^p, \quad x \in \partial\mathbb{R}^n_+,$$

where $\mathbb{R}^n_+ = \{(x_1, x_2, \cdots, x_n), x_1 > 0\}$, prove the following theorem.

Theorem 6.11. *(i) If* $1 < p < n/(n-2)$, *then* $u \equiv 0$.
(ii) If $p = n/(n-2)$, *then*

$$u(x) = \frac{c_1}{|x - x_0|^{n-2}} \quad \textit{in } \mathbb{R}^n_+,$$

for some $x_0 \notin \mathbb{R}^n_+$, *and some constant* $c_1 \geqslant 0$.

Chapter 7
Blow-Up Rate

It is established in Chap. 5 that the nonlinearity causes the blow-up to occur at a finite time in certain situations. If the solution to the ODE

$$u_t = f(u),$$

blows up at a finite time $t = T$ with $u(T - 0) = +\infty$, then

$$u = G(T - t),$$

where $G(\xi)$ is the inverse function of $\displaystyle\int_u^\infty \frac{d\eta}{f(\eta)}$. In particular, if $f(u) = u^p$ $(p > 1)$, the *blow-up rate* for this ODE is $(T - t)^{1/(p-1)}$.

Definition 7.1. If $G(s) \nearrow +\infty$ as $s \searrow 0$ and

$$G(c_2(T-t)) \leqslant \|u\|_{L^\infty} \leqslant G(c_1(T-t)) \quad \text{for } 0 < T - t \ll 1,$$

for some $c_2 > c_1 > 0$, we say $G(c(T-t))$ is the (L^∞) blow-up rate.

A natural question is whether the blow-up rate for the PDE remains the same as that for the corresponding ODE. In other words, is the diffusion strong enough to have an impact on the blow-up rate? The answer depends on the system. The PDE blow-up rate *can be different* from the ODE blow-up rate for some equations. But for a large class of equations, the rate remains the same.

Remark 7.1. Even if the ODE blow-up rate and PDE blow-up rate are the same, the constants in the definition of the blow-up rate can be different in the case of nonlinear principle part. Namely, suppose that the ODE solution is given by $G(T - t)$, the PDE solution at the blow-up point can behave like $G(c(T - t))$ with $c < 1$.

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7.1 Blow-Up Rate Lower Bound for Internal Heat Source

The lower bound is the easier part of the blow-up rate estimate. We begin by considering the classical solution to the equation

$$u_t - \Delta u = f(u), \quad x \in \Omega,\ t > 0, \tag{7.1}$$

$$u = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{7.2}$$

$$u = u_0(x) \geqslant 0, \tag{7.3}$$

where we assume that Ω is a bounded domain, $u \geqslant 0$ and blows up at a finite time T, and $f(u) > 0$ for $u > 0$. Notice that the assumption of finite time blow-up implies that $\displaystyle\int^{\infty} \frac{du}{f(u)} < \infty$.

Theorem 7.1.

$$\max_{x \in \overline{\Omega}} u(x,t) \geqslant G(T-t) \quad \text{for } 0 < T - t \ll 1, \tag{7.4}$$

where $G(\xi)$ is the inverse function of $\displaystyle\int_u^{\infty} \frac{d\eta}{f(\eta)}$.

Proof. Define

$$M(t) = \max_{x \in \overline{\Omega}} u(x,t).$$

The function $M(t)$ is clearly Lipschitz continuous and therefore $M'(t)$ exists almost everywhere. For each $t \in (0,T)$, the maximum of $u(\cdot,t)$ must be attained at an interior point (x^t,t), on which we must have $\Delta u(x^t,t) \leqslant 0$. Clearly,

$$u(x^{t_0},t) \leqslant M(t),\ \forall t \in (0,T), \quad u(x^{t_0},t_0) = M(t_0).$$

Therefore $M'(t_0) = u_t(x^{t_0},t_0)$ at any t_0 where M is differentiable. It follows that

$$M'(t) = u_t(x^t,t) \leqslant f(u(x^t,t)) = f(M(t)) \quad a.e.\ t \in (0,T),$$

which implies

$$\int_{M(t)}^{\infty} \frac{d\eta}{f(\eta)} \leqslant T - t. \qquad \square$$

Remark 7.2. We established that the ODE blow-up rate is the lower bound for the PDE blow-up rate. Other than the regularity assumptions and necessary conditions for blow-up, there are essentially *no other growth assumptions or monotonicity assumptions.*

Remark 7.3. If $f(u) = u^p$, then

$$\max_{x \in \overline{\Omega}} u(x,t) \geqslant \left(\frac{1}{p-1}\right)^{1/(p-1)} \frac{1}{(T-t)^{1/(p-1)}} \qquad \text{for } 0 < T - t \ll 1;$$

if $f(u) = \exp u$, then

$$\max_{x\in\overline{\Omega}} u(x,t) \geqslant \log \frac{1}{T-t} \quad \text{for } 0 < T-t \ll 1;$$

and if $f(u) = u \ \log^p(u+1)$, then

$$\log\Big(\max_{x\in\overline{\Omega}} u(x,t)+1\Big) \geqslant \Big(\frac{1}{p-1}\Big)^{1/(p-1)} \frac{1}{(T-t)^{1/(p-1)}} \quad \text{for } 0 < T-t \ll 1.$$

7.2 Blow-Up Rate Lower Bound: A Scaling Method

Consider the classical solution to the equation

$$u_t - \Delta u = 0, \qquad x \in \Omega,\ t>0, \tag{7.5}$$

$$\frac{\partial u}{\partial n} = f(u), \quad x \in \partial\Omega,\ t>0, \tag{7.6}$$

$$u(x,0) = u_0(x) \geqslant 0, \tag{7.7}$$

where we assume that Ω is a bounded domain,

$$u \geqslant 0,\ x \in \overline{\Omega},\ t>0, \quad \text{and blows up at a finite time } T, \tag{7.8}$$

$$f \in C^1, \quad f(u) > 0 \quad \text{for } u > 0. \tag{7.9}$$

The lower bound for the case of boundary source is not as simple as the case of interior source, since one cannot directly compare u_t with the source term. As a matter of fact, $f(u)$ is related to the first order spatial derivatives while u_t is related to the second order spatial derivatives.

For the problems with nonlinear boundary source, in order to get the "correct" order of growth for u_t, we use the scaling method (c.f. Hu–Yin [75]), plus the discretization method (c.f. Hu [73]) (the papers [73, 75] actually deals only with $f(u) = u^p$, but the method clearly extends to the general $f(u)$, as given here).

We begin with a lemma modified from [73, Lemma 3.1].

Lemma 7.2 (Doubling). *Suppose that there exist* $c_1 > 0$ *and a function* $k(u)$ *such that*

$$k'(u) \geqslant 0 \quad \textit{for } u \geqslant c_0 > 0, \qquad k(c_0) > 0,$$

$$t_j < T, \quad \textit{and } t_j \nearrow T \quad \textit{as } j \to \infty,$$

$$M(t_j) \to +\infty \quad \textit{as } j \to \infty,$$

$$k(M(t_{j+1})) = 2k(M(t_j)) > 0, \quad j = 0, 1, 2, \cdots,$$
$$k(M(t_j))\frac{t_{j+1} - t_j}{M(t_{j+1}) - M(t_j)} \geqslant \quad (\leqslant) \quad c_1, \qquad j = 0, 1, 2, \cdots.$$

Then there exists a constant $C > 0$*, independent of* t_0*, such that*

$$\int_{M(t_0)}^{\infty} \frac{d\eta}{k(\eta)} \leqslant \quad (\geqslant) \quad C(T - t_0).$$

Proof. By our assumptions,

$$t_{j+1} - t_j \geqslant c_1 \frac{M(t_{j+1}) - M(t_j)}{k(M(t_j))} \geqslant c_1 \int_{M(t_j)}^{M(t_{j+1})} \frac{d\eta}{k(\eta)}.$$

Taking summation over $j = 0$ through ∞, we obtain

$$T - t_0 \geqslant c_1 \int_{M(t_0)}^{\infty} \frac{d\eta}{k(\eta)}.$$

Similarly, in the case the third inequality is reversed in the assumptions,

$$t_{j+1} - t_j \leqslant 2c_1 \frac{M(t_{j+1}) - M(t_j)}{k(M(t_{j+1}))} \leqslant 2c_1 \int_{M(t_j)}^{M(t_{j+1})} \frac{d\eta}{k(\eta)}.$$

Taking summation over $j = 0$ through ∞, we derive

$$T - t_0 \leqslant 2c_1 \int_{M(t_0)}^{\infty} \frac{d\eta}{k(\eta)}. \qquad \square$$

Theorem 7.3. *Let the assumptions (7.8) and (7.9) be in force and that* $\partial\Omega \in C^{0,1}$ *(i.e., uniformly Lipschitz). Assume without loss of generality that* $u_0(x) \geqslant c_0 > 0$*. Set*

$$M(t) = \max_{x \in \overline{\Omega}} u(x,t), \qquad k(u) = \sup_{c_0 < \eta \leqslant u} \frac{f^2(\eta)}{\eta}.$$

We further assume that k *satisfies, for some* $L \gg 1$ *and some* $\beta > 0$

$$\frac{k(\xi_2)}{k(\xi_1)} = 2, \quad \xi_2 \geqslant \xi_1 \geqslant L \quad \textit{implies} \quad \frac{\xi_2}{\xi_1} \geqslant 1 + \beta. \tag{7.10}$$

Then there exists $C > 0$ *such that*

$$\int_{M(t)}^{\infty} \frac{d\eta}{k(\eta)} \leqslant C(T - t).$$

Proof. Since $M(t)$ may not be monotone in t, for each t^* we let v to be the solution of

$$\begin{aligned} v_t - \Delta v &= 0, \qquad x \in \Omega,\ t > t^*, \\ \frac{\partial v}{\partial n} &= f_1(v), \quad x \in \partial\Omega,\ t > t^*, \\ v(x,t^*) &= M(t^*), \end{aligned}$$

where $f_1(v) = \sqrt{vk(v)}$. It is clear that $f_1'(v) \geqslant 0$ and

$$f_1(v) \geqslant \sqrt{v\frac{f^2(v)}{v}} \geqslant f(v) \quad \text{for } v \geqslant c_0.$$

Thus v is monotone in t and by the comparison principle,

$$u(x,t) \leqslant v(x,t) \quad \text{for } t > t^*.$$

In particular, $v(x,t)$ must blow up in finite time. Take t_1 such that

$$M_1 = \max_{x\in\overline{\Omega}} v(x,t_1), \qquad k(M_1) = 2k(M(t^*)). \tag{7.11}$$

Then clearly, $M_1 \geqslant M(t_1)$ and there exists a $\widetilde{t}_1 \geqslant t_1$ such that

$$M_1 = M(\widetilde{t}_1). \tag{7.12}$$

Set

$$\lambda = \frac{M_1}{f_1(M_1)}.$$

Define the scaling

$$w(y,s) = \frac{1}{M_1} v(\lambda y + x^*, \lambda^2 s + t_1).$$

Then

$$\begin{aligned} w_s - \Delta_y w &= 0, \quad y \in \Omega_\lambda,\ \frac{t^* - t_1}{\lambda^2} < s < 0, \\ \frac{\partial w}{\partial n_y} &= \frac{f_1(M_1 w)}{f_1(M_1)}, \quad y \in \partial\Omega_\lambda,\ \frac{t^* - t_1}{\lambda^2} < s < 0, \\ w(0,0) &= 1, \\ 0 \leqslant w &\leqslant 1, \quad y \in \Omega_\lambda,\ \frac{-t_1}{\lambda^2} < s < 0, \end{aligned}$$

where $\Omega_\lambda = \{y;\ \lambda y + x^* \in \Omega\}$. Since the function $g(w) := \dfrac{f_1(M_1 w)}{f_1(M_1)}$ is uniformly bounded for $s < 0$, we have the Hölder estimates (Remark 3.5) for w:

$$|w(0,s_1) - w(0,s_2)| \leqslant C|s_1 - s_2|^{\gamma/2} \quad \text{for } -\min\left(1, \frac{t_1 - t^*}{\lambda^2}\right) \leqslant s_1 < s_2 \leqslant 0. \tag{7.13}$$

Taking $s_2 = 0$, $s_1 = -\min\left(1, \dfrac{t_1 - t^*}{\lambda^2}\right)$, we conclude

$$C\left|\frac{t_1 - t^*}{\lambda^2}\right|^{\gamma/2} \geqslant 1 - \frac{M(t^*)}{M_1} \tag{7.14}$$

(in the case $(t_1 - t^*)/\lambda^2 > 1$, (7.14) is obviously true; on the other hand in the case $(t_1 - t^*)/\lambda^2 \leqslant 1$, we have $s_1 = -(t_1 - t^*)/\lambda^2$ and we obtain (7.14) from the definition of w).

Assume without loss of generality that $M(t^*) > L$, then the assumption (7.10) implies that (using also (7.11))

$$\frac{M_1}{M(t^*)} \geqslant 1 + \beta. \tag{7.15}$$

Substituting (7.15) into (7.14), we derive

$$\frac{t_1 - t^*}{\lambda^2} \geqslant c_0 > 0 \qquad (c_0 \text{ is independent of } t^*),$$

i.e.,

$$\frac{k(M_1)}{M_1}(t_1 - t^*) \geqslant c_0. \tag{7.16}$$

Recalling (7.12), (7.15) and the fact that $\widetilde{t}_1 \geqslant t_1$, we deduce from (7.16) that

$$\frac{k(M(\widetilde{t}_1))}{M(\widetilde{t}_1) - M(t^*)}(\widetilde{t}_1 - t^*) \geqslant \frac{k(M_1)}{M_1}(t_1 - t^*) \geqslant c_0. \tag{7.17}$$

Using $\widetilde{t}_1$ as the new starting point t^*, we can now proceed to obtain $\widetilde{t}_2$ such that

$$\frac{k(M(\widetilde{t}_2))}{M(\widetilde{t}_2) - M(\widetilde{t}_1)}(\widetilde{t}_2 - \widetilde{t}_1) \geqslant c_0. \tag{7.18}$$

This procedure enables us to obtain a sequence of $\widetilde{t}_j$ satisfying the assumptions of Lemma 7.2, and we conclude the theorem by applying Lemma 7.2. □

Remark 7.4. If

$$\xi k'(\xi) \leqslant C k(\xi) \quad \text{for } \xi \gg 1,$$

then clearly (7.10) is satisfied.

Remark 7.5. In the scaling method, the key for deriving a blow-up rate lower bound is to establish (7.16). Note that this is equivalent to saying the scaled solution w cannot jump from value $1/(\beta+1)$ to value 1 in a time interval that may shrink to 0. Thus the lower bound blow-up rate estimate is essentially a modulus of continuity regularity estimate.

As an immediate corollary, we have

Corollary 7.4 (Necessary condition). *Let Ω be a bounded Lipschitz domain. Suppose that (7.10) holds, and that for some $c_0 > 0$,*

$$f \in C^1, \quad \frac{d}{du}\left[\frac{f^2(u)}{u}\right] \geqslant 0 \quad \textit{for } u > c_0, \tag{7.19}$$

and for some initial datum $u_0 \geqslant 0$, the solution $u(x,t)$ blows up at a finite time T, then

$$\int_{c_0}^{\infty} \frac{\eta d\eta}{f^2(\eta)} < +\infty. \tag{7.20}$$

Remark 7.6. This necessary condition, under the assumptions (7.10) and (7.19), is also ***sufficient*** for all nontrivial solutions to blow up in finite time (see Theorem 7.14). We leave its proof as an exercise.

We already established in Chap. 5 that, in the case $f(u) = u^p$ in (7.6), all nontrivial solutions blow-up in finite time. We now state the theorem for the lower bound.

Corollary 7.5. *Let Ω be a bounded Lipschitz domain. If in (7.6), $f(u) = u^p$ $(p > 1)$, then there exists $c_0 > 0$ such that*

$$\max_{x \in \overline{\Omega}} u(x,t) \geqslant \frac{c_0}{(T-t)^{1/[2(p-1)]}},$$

where T is the blow-up time.

The function

$$f(u) = u \ \log^p(u+1), \qquad (p > 1/2),$$

also satisfies the assumptions of Theorem 7.3, in this case we have

Corollary 7.6. *Let Ω be a bounded Lipschitz domain and $f(u) = u \ \log^p(u+1)$ $(p > 1/2)$ in (7.6). Then there exists $c_0 > 0$ such that*

$$\log\left(\max_{x \in \overline{\Omega}} u(x,t) + 1\right) \geqslant \frac{c_0}{(T-t)^{1/(2p-1)}}, \tag{7.21}$$

where T is the blow-up time.

Remark 7.7. Here we only use Hölder estimates. If Ω and $f(u)$ is more regular, say $\partial\Omega \in C^{2+\alpha}$, $f(u) = u^p$, then the Schauder estimate (Theorem 3.4) can be used, and in this case discretization is not necessary. We can estimate u_t directly in terms of the function $k(u)$ through a scaling and applying the Schauder estimates.

7.3 Blow-Up Rate Upper Bound: Friedman–McLeod's Method

Consider the equation

$$u_t - \Delta u = f(u), \quad x \in \Omega,\ t > 0, \tag{7.22}$$

$$u = 0, \quad x \in \partial\Omega,\ t > 0, \tag{7.23}$$

$$u = u_0(x) \geqslant 0. \tag{7.24}$$

We assume that the solution u blows up in finite time T and want to find out the blow-up rate. Roughly speaking, the ODE rate is the same as the PDE rate if and only if the diffusion term Δu *cancels* only a fraction of the source term $f(u)$. In another words, u_t is still roughly proportional to $f(u)$ for large u in this situation. Friedman–McLeod [42] rigorously established this by using the maximum principle through carefully constructed auxiliary functions.

Theorem 7.7. *Consider* (7.1)–(7.3)*, where we assume that* Ω *is a smooth and bounded domain,* $u_0 \in C^2(\overline{\Omega})$*,*

$$u_0 \geqslant 0, \quad \Delta u_0(x) + f(u_0(x)) \geqslant 0,\ x \in \Omega, \qquad u_0(x) = 0,\ x \in \partial\Omega. \tag{7.25}$$

We further assume that

$$f \in C^1, \quad f(u) > 0 \quad \text{for } u > 0, \tag{7.26}$$

and that there exists some function F *such that*

$$F \geqslant 0,\ F' \geqslant 0,\ F'' \geqslant 0, \quad f'F - fF' \geqslant 0. \tag{7.27}$$

Finally, we assume that the blow-up set is a compact subset of Ω*. Under these assumptions, for any small* $\eta > 0$ *there exists a* $\delta > 0$ *such that*

$$u_t \geqslant \delta F(u) \quad \text{in } \Omega^\eta \times (\eta, T), \tag{7.28}$$

where

$$\Omega^\eta = \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) > \eta\}.$$

Proof. Following Sect. 5.3 of Chap. 5, we find that $u(x,t) \geqslant u_0(x)$ and then, as in the argument (5.25), we conclude $u(x,t+\eta) \geqslant u(x,t)$ for any $\eta > 0$. This implies that (noticing that $u_t \not\equiv 0$)

$$u_t(x,t) > 0 \quad \text{for } x \in \Omega,\ 0 < t < T. \tag{7.29}$$

The function

$$J = u_t - \delta F(u)$$

satisfies

$$J_t - \Delta J - f'(u)J = \delta(f'F - fF') + \delta F''|\nabla u|^2 \geqslant 0,$$

where (7.27) is used.

In view of the assumption that the blow-up set is a compact set, we have for small $\eta > 0$,

$$F(u) \leqslant C_0 < \infty \quad \text{if } x \in \partial\Omega^\eta,\ 0 < t < T.$$

Using also (7.29), we find that

$$J \geqslant 0 \quad \text{on } \Big(\partial\Omega^\eta \times (0,T)\Big) \cup \Big(\Omega^\eta \times \{t = \eta\}\Big),$$

provided we first take $\delta > 0$ to be small. Now the theorem follows from the maximum principle. □

To use this theorem, however, we need to establish the fact that blow-up cannot occur at the boundary. Intuitively this looks like to be true, since $u = 0$ on the boundary. But we need to exclude the possibility of a sequence of points x^k approaching the boundary on which the solution u becomes unbounded as t approaches T.

We begin with following lemma:

Lemma 7.8. *In addition to the assumption* (7.26)*, assume that*

$$\Omega \text{ is strictly convex with } \partial\Omega \in C^2.$$

Then for any point $x_0 \in \partial\Omega$, any $c_0 > 0$, and the exterior unit normal $\gamma = \mathbf{n}(x_0)$, there exists a $\eta > 0$ such that

$$\frac{\partial u}{\partial \gamma}(x,t) < 0 \quad \text{for } x \in B_\eta(x_0) \cap \overline{\Omega},\ c_0 < t < T, \tag{7.30}$$

$$\frac{\partial u}{\partial \gamma}(x,t) < -\eta \quad \text{for } x \in B_\eta(x_0) \cap \partial\Omega,\ c_0 < t < T, \tag{7.31}$$

Proof. We can first establish the estimate at $t = c_0 > 0$ by Hopf's lemma. Then we can use reflection along a plane similar to the method used in Chap. 6 to establish (7.30). The detail of the proof is left as an Exercise.

To prove (7.31), we can apply the maximum principle to derive $u(x,t) \geqslant v(x,t)$, where $v(x,t)$ is the solution of $v_t - \Delta v = 0$ in $\Omega \times (0,T)$, $v = 0$ on $\partial\Omega \times (0,T)$,

$v(x, c_0/2) = u(x, c_0/2)$. Since v clearly satisfies (7.31), and $\frac{\partial u}{\partial \gamma}(x,t) \leqslant \frac{\partial v}{\partial \gamma}(x,t)$ for $x \in B_\eta(x_0) \cap \partial\Omega$, we conclude (7.31). □

Lemma 7.9. *In addition to the assumptions of Lemma 7.8, assume that there exists a small $\delta > 0$ and some function F such that*

$$F \geqslant 0,\ F' \geqslant 0,\ F'' \geqslant 0, \quad f'F - fF' + 2F \geqslant 2\delta FF', \quad \int_\delta^\infty \frac{d\eta}{F(\eta)} < \infty. \tag{7.32}$$

Then the blow-up set is a compact subset of Ω.

Proof. Take $x_0 \in \partial\Omega$. By a rotation and a translation if necessary, we assume that the positive x_n-axis is the exterior normal direction and $x_0 = 0$. By using a finite covering if necessary, we can find a small $\lambda > 0$ such that

$$\frac{\partial u}{\partial x_n}(x,t) < 0 \quad \text{for } x_n \geqslant -\lambda,\ x \in \Omega,\ c_0 < t < T, \tag{7.33}$$

$$\frac{\partial u}{\partial x_n}(x,t) < -\delta \quad \text{for } x_n > -\lambda,\ x \in \partial\Omega,\ c_0 < t < T. \tag{7.34}$$

Let

$$J = -\frac{\partial u}{\partial x_n}(x,t) - \delta(x_n + \lambda)^2 F(u).$$

Then by (7.33) and (7.34), for sufficiently small $\delta > 0$,

$$J > 0 \qquad \text{on } \partial\Big(\{x \in \Omega, x_n > -\lambda\}\Big) \times \{c_0 < t < T\} \bigcup \{x \in \Omega, x_n > -\lambda\}) \times \{t = c_0\}.$$

A direct computation shows that, for small $\delta > 0$,

$$\begin{aligned} &J_t - \Delta J - f'(u)J + 4\delta(x_n + \lambda)F'J \\ &= \delta(x_n + \lambda)^2\Big(f'F - fF' + F''|\nabla u|^2 - 4\delta(x_n + \lambda)FF' + 2\frac{F}{(x_n + \lambda)^2}\Big) \\ &\geqslant 0 \quad \text{in } \{x \in \Omega, x_n > -\lambda\} \times \{c_0 < t < T\}. \end{aligned}$$

Thus $J > 0$ in $\{x \in \Omega, x_n > -\lambda\} \times \{c_0 < t < T\}$ by the maximum principle.

Now fixing $(x_1, \cdots, x_{n-1})$ and integrating over $x_n \in [-\lambda, -\lambda/2]$, we obtain

$$\int_{u(0,\cdots,0,-\lambda/2)}^\infty \frac{d\eta}{F(\eta)} \geqslant \int_{u(0,\cdots,0,-\lambda/2)}^{u(0,\cdots,0,-\lambda)} \frac{d\eta}{F(\eta)} > \frac{\delta}{24}\lambda^3.$$

Thus $u(x_1, \cdots, x_{n-1}, -\lambda/2)$ is bounded above in a small neighborhood of x_0. Since $u(x_1, \cdots, x_{n-1}, x_n) \leqslant u(x_1, \cdots, x_{n-1}, -\lambda/2)$ for $-\lambda/2 < x_n$, and since we can follow this procedure for any direction, we can use a finite covering argument to conclude the lemma. □

Corollary 7.10. *If Ω is a bounded convex domain with $\partial\Omega \in C^2$ and either $f(u) = u^p$ $(p > 1)$ or $f(u) = \exp u$, then the blow-up set is a compact subset of Ω.*

Proof. It suffices to find an F satisfying (7.32). In the case $f(u) = u^p$, we can take $F(u) = u^q$ for any $1 < q < p$. In the case $f(u) = \exp u$, we can take $F(u) = \exp(u/2)$. □

Corollary 7.11. *Let Ω be a bounded convex domain with $\partial\Omega \in C^2$. Assume that $u_0 \in C^2(\overline{\Omega})$,*

$$u_0 \geqslant 0, \quad \Delta u_0(x) + f(u_0(x)) \geqslant 0,\ x \in \Omega, \qquad u_0(x) = 0,\ x \in \partial\Omega.$$

If $f(u) = u^p$ and T is the blow-up time, then

$$u(x,t) \leqslant \frac{C}{(T-t)^{1/(p-1)}}, \tag{7.35}$$

and if $f(u) = \exp u$ and T is the blow-up time, then

$$u(x,t) \leqslant \log \frac{1}{T-t} + C.$$

Proof. By Corollary 7.10, the blow-up set is a compact subset of Ω. Now we choose $F \equiv f$ in Theorem 7.7 to finish the proof. □

Remark 7.8. The assumption in Theorem 7.7 guarantees that u_t is nonnegative. This is essential in the proof. Because of this restriction, this proof *cannot* be applied to solutions oscillating in the t direction.

Remark 7.9. One might ask whether it is true that the ODE blow-up rate (*Type I*) is always the same as the PDE blow-up rate. For the case $\Omega = \mathbb{R}^n$ $(n \geqslant 11)$,

$$f(u) = u^p, \qquad p > p^* := \frac{n - 2\sqrt{n-1}}{n - 4 - 2\sqrt{n-1}},$$

it is shown in Herrero–Velazquez [71] there exists positive solutions $u(r,t)$ (see also Mizoguchi [106]) such that

$$\limsup_{t \nearrow T} (T-t)^{1/(p-1)} u(0,t) = +\infty,$$

such a rate is called *Type II* blow-up. Here p^* is the *Joseph–Lundgren constant [79].*

It is also established in Filippas–Herrero–Velazquez [39] that type II blow-up can also occur for $p = (n+2)/(n-2)$.

On the other hand, in the case that $\Omega = B_R$ in (7.22)–(7.24) $(n \geqslant 3)$ and

$$f(u) = |u|^{p-1}u, \qquad \frac{n+2}{n-2} < p < p^*,$$

where

$$p^* = \frac{n - 2\sqrt{n-1}}{n - 4 - 2\sqrt{n-1}} \quad \text{if } n \geqslant 11, \quad p^* = \infty \quad \text{if } 3 \leqslant n < 11.$$

It is shown in Matano–Merle [102] that any radially symmetric blow-up solution (the solution is allowed to change sign) must satisfy

$$\max_{x \in \overline{\Omega}} |u(x,t)| \leqslant \frac{C}{(T-t)^{1/(p-1)}},$$

such a rate is called *Type I* blow-up.

Remark 7.10. The Friedman–McLeod method compares u_t with the source term $f(u)$, which is intuitively promising since these terms appear at the "same level" in the equation. It is therefore not clear whether this method can be applied to equations with boundary source. In the next section, we shall introduce the scaling method for dealing with equations with boundary source.

Remark 7.11. Under the assumption which makes $u_t \geqslant 0$, the estimate (7.35) is valid for all $p \in (1, \infty)$. Using similarity variables, Giga–Kohn [65] established (7.35) for $1 < p < (n+2)/(n-2)$ for convex domains without the assumption that guarantees that $u_t \geqslant 0$. We shall further discuss this in Chap. 8.

Remark 7.12. With reasonable assumptions on f, the function

$$F(u) = \frac{1}{2\delta} \frac{f(u)}{\ln f(u)}$$

satisfies (7.32). It is optimal in the sense that all "leading order" terms are taken care of and the fourth inequality in (7.32) will be violated for sufficiently large u if the coefficient $\frac{1}{2\delta}$ is replaced by $\frac{1}{2\delta} + \varepsilon$ for any small $\varepsilon > 0$. In the case of radially symmetric solutions with $f(0) = 0$, the argument in Lemma 7.9 (with $\delta = 1$) leads to

$$u_r + rF(u) \leqslant 0.$$

This estimate is then used to derive sharp estimates containing "log" terms in blow-up behaviors. The Chaps. 9 and 10 therein contain detailed analysis when $f(u) = u(\ln u)^p$. We refer to Galaktionov–Vàzquez [58] for more details.

7.4 Blow-Up Rate Upper Bound: A Scaling Method

The scaling method can also be used to establish the upper bound for the blow-up rate. However, the proof is more demanding in this case as explained in Remark 7.13. Consider the equation

$$u_t - \Delta u = 0, \quad x \in \Omega,\ t > 0, \tag{7.36}$$

$$\frac{\partial u}{\partial n} = f(u), \quad x \in \partial\Omega,\ t > 0, \tag{7.37}$$

$$u = u_0(x) \geqslant 0, \tag{7.38}$$

where we assume that

$$\Omega \text{ is a bounded Lipschitz domain}, \tag{7.39}$$

$$f \in C^1, \quad f(u) > 0 \quad \text{for } u > 0, \tag{7.40}$$

$$\limsup_{t \nearrow T} \max_{x \in \overline{\Omega}} u(x,t) = +\infty, \tag{7.41}$$

and that the function

$$k(u) = \sup_{c_0 < \eta < u} \frac{f^2(\eta)}{\eta} \tag{7.42}$$

satisfies (7.10).

For this problem with boundary source, we shall use the again scaling method (c.f. Hu–Yin [75]) and the discretization (c.f. Hu [73]) to establish an upper bound. We will first introduce the method, and then impose assumptions in addition to (7.10), (7.39)–(7.41) to carry out the procedure. Let

$$M(t) = \max_{0 \leqslant \tau \leqslant t} \max_{x \in \overline{\Omega}} u(x,\tau).$$

Then $M(t) \nearrow \infty$ as $t \nearrow T$. Fix t^* such that $0 < T - t^* \ll 1$ and take $\widehat{x} \in \partial\Omega$, $\widehat{t} \in (T/2, t^*]$ such that

$$M(t^*) = u(\widehat{x}, \widehat{t}). \tag{7.43}$$

Since $M(t) \to \infty$ and $k(u)$ is monotonically increasing, there is a first time $t_1 > t^*$ such that

$$\begin{aligned} k(M(t_1)) &= 2k(M(t^*)), \\ \max_{x \in \overline{\Omega}} u(x,t_1) &= M(t_1). \end{aligned} \tag{7.44}$$

Note that (7.10) implies that, for $\beta > 0$,

$$\frac{M(t_1)}{M(t^*)} \geqslant 1 + \beta. \tag{7.45}$$

Define the scaling

$$w(y,s)=\frac{1}{M(t_1)}u(\lambda y+\widehat{x},\lambda^2 s+\widehat{t}\,),\quad y\in\Omega_\lambda,\ \frac{-\widehat{t}}{\lambda^2}<s<\frac{t_1-\widehat{t}}{\lambda^2},$$

where $\Omega_\lambda=\{y;\ \lambda y+\widehat{x}\in\Omega\}$, and

$$\lambda=\sqrt{\frac{M(t_1)}{k(M(t_1))}}. \tag{7.46}$$

Then

$$w_s-\Delta_y w=0,\quad y\in\Omega_\lambda,\ -\frac{\widehat{t}}{\lambda^2}<s<\frac{t_1-\widehat{t}}{\lambda^2}, \tag{7.47}$$

$$\frac{\partial w}{\partial n_y}=\frac{f(M(t_1)w)}{\sqrt{M(t_1)k(M(t_1))}},\quad y\in\partial\Omega_\lambda,\ -\frac{\widehat{t}}{\lambda^2}<s<\frac{t_1-\widehat{t}}{\lambda^2}, \tag{7.48}$$

$$0<w(y,s)\leqslant 1,\quad y\in\Omega_\lambda,\ -\frac{\widehat{t}}{\lambda^2}<s<\frac{t_1-\widehat{t}}{\lambda^2}, \tag{7.49}$$

$$w(0,0)=\frac{M(t^*)}{M(t_1)}\leqslant\frac{1}{1+\beta},\quad \max_{y\in\overline{\Omega}_\lambda} w\Big(y,\frac{t_1-\widehat{t}}{\lambda^2}\Big)=1. \tag{7.50}$$

It follows from the definition of $k(u)$ and the fact that $w\leqslant 1$ that

$$0\leqslant\frac{f(M(t_1)w(y,s))}{\sqrt{M(t_1)k(M(t_1))}}\leqslant 1,\quad y\in\partial\Omega_\lambda,\ -\frac{\widehat{t}}{\lambda^2}<s<\frac{t_1-\widehat{t}}{\lambda^2}. \tag{7.51}$$

The upper bound rate estimate depends on the following estimate:

$$s_\lambda:=\frac{t_1-\widehat{t}}{\lambda^2}\leqslant C^*, \tag{7.52}$$

for some $C^*>1$ independent of λ, t^*, $\widehat{t}$. Extra assumptions are necessary in order to derive this estimate.

If we established (7.52), then

$$\begin{aligned}k(M(t_1))\frac{t_1-t^*}{M(t_1)-M(t^*)}&=\frac{1}{1-M(t^*)/M(t_1)}k(M(t_1))\frac{t_1-t^*}{M(t_1)}\\&\leqslant\frac{1+\beta}{\beta}\,\frac{t_1-\widehat{t}}{\lambda^2}\leqslant C^*\frac{1+\beta}{\beta}.\end{aligned}$$

Using t_1 as the new starting point in place of t^*, we can repeat the above procedure. Thus we can obtain a sequence $t_j\to T$ satisfying the assumptions of Lemma 7.2, and thus

$$\int_{M(t^*)}^{\infty} \frac{d\eta}{k(\eta)} \geqslant \delta(T-t), \tag{7.53}$$

for some $\delta > 0$.

Remark 7.13. Since $\displaystyle\int^{\infty} \frac{d\eta}{k(\eta)}$ is finite, $k(\eta)$ must grow more than u, at least on a subsequence. With very mild additional assumptions on the function $k(\eta)$, "$k(\xi_2) = 2k(\xi_1)$ for $\xi_2 > \xi_1 \gg 1$" would imply $\xi_2/\xi_1 < C$ for some constant $C > 1$. In this case we have

$$w(0,0) = \frac{M(t^*)}{M(t_1)} \geqslant c_0.$$

If we use the scaling method, the key for deriving a blow-up rate upper bound is to establish (7.52). As indicated in Remark 7.5, the lower bound is essentially reduced to regularity estimates. However, (7.52) is basically equivalent to saying that the scaled solution w should take a finite time interval (uniformly bounded) to go from value in $[c_0, 1-c_0]$ ($c_0 > 0$) to value 1. Is clear that a regularity estimate alone is not enough.

Now we will give various conditions to ensure (7.52).

Theorem 7.12. *Consider the equation*

$$u_t - \Delta u = 0, \quad x \in \Omega,\ t > 0, \tag{7.54}$$

$$\frac{\partial u}{\partial n} = f(u), \quad x \in \partial\Omega,\ t > 0, \tag{7.55}$$

$$u = u_0(x) \geqslant 0, \tag{7.56}$$

where we assume that

$$\Omega \text{ is a bounded domain with } \partial\Omega \in C^{1+\alpha}, \tag{7.57}$$

$$f \in C^1, \quad f(u) > 0 \quad \text{for } u > 0, \tag{7.58}$$

$$\lim_{\eta\to\infty} \frac{f(\eta)}{\eta^p} = 1, \quad 1 < p < \frac{n}{n-2} \text{ if } n \geqslant 3, 1 < p < \infty \text{ if } n = 1,2, \tag{7.59}$$

$$u_0 \in C^2(\overline{\Omega}), \quad -\Delta u_0(x) \leqslant 0 \text{ in } \Omega,\ \frac{\partial u_0(x)}{\partial n} \leqslant f(u_0(x)) \text{ on } \partial\Omega. \tag{7.60}$$

Then

$$\max_{x\in\overline{\Omega}} u(x,t) \leqslant \frac{C}{(T-t)^{1/[2(p-1)]}}, \tag{7.61}$$

where T is the blow-up time.

Proof. It is not difficult to verify that (7.59) implies (7.10). The assumption (7.59) also implies that the scaled function $w(y,s)$ satisfies

$$w(0,0) \geqslant c_0 > 0, \tag{7.62}$$

for some c_0 independent of t^*, and

$$\frac{f(\eta w)}{\sqrt{\eta k(\eta)}} \to w^p \quad \text{uniformly for } w \in [0,1] \text{ as } \eta \to +\infty. \tag{7.63}$$

Following the proof of Theorem 5.4 it is not difficult to prove that

$$u_t(x,t) \geqslant 0 \quad \text{for } x \in \overline{\Omega},\ 0 < t < T. \tag{7.64}$$

It follows that the scaled function $w(y,s)$ satisfies $w_s(y,s) \geqslant 0$ in addition to (7.47)–(7.50).

If (7.52) is not true, then we can find a sequence $t_j^* \nearrow T$ such that, for $(t_1)_j$ defined as the doubling time for the function $k(\eta)$ from $\eta = M(t_j^*)$ and the corresponding λ_j is defined by (7.46), we have $s_{\lambda_j} \to +\infty$. We denote the corresponding scaled function by $w_j(y,s)$. We now make a rotation so that the exterior normal at $y = 0$ is $(1, 0, \cdots, 0)$, the solution is still denoted by $w_j(y,s)$.

By Remark 3.5 and the compactness $C^{\gamma,\gamma/2}(\overline{\Omega}_{\lambda_j} \cap \{|y| \leqslant K\}) \times [-K,K]) \hookrightarrow C(\overline{\Omega}_{\lambda_j} \cap \{|y| \leqslant K\}) \times [-K,K])$ (Ascoli–Arzelà theorem), we can take a further subsequence if necessary to obtain

$$\begin{aligned} &\|w_j - \psi\|_{C((\overline{\Omega}_{\lambda_j} \cap \{|y| \leqslant K\}) \times [-K,K])} \to 0, \\ &\nabla_y w_j\, \chi_{\Omega_{\lambda_j}}(y) \rightharpoonup \nabla_y \psi \quad \text{in } \{L^2((\mathbb{R}^n_+ \cap \{|y| \leqslant K\}) \times [-K,K])\}^n. \end{aligned}$$

We can further use a diagonalization procedure so that the above convergence is valid for any $K > 1$. It follow that $\psi \in C(\overline{\mathbb{R}^n_+} \times (-\infty,\infty))$ is a weak solution of

$$\psi_s - \Delta_y \psi = 0, \quad y \in \mathbb{R}^n_+,\ -\infty < s < \infty, \tag{7.65}$$

$$\frac{\partial \psi}{\partial n_y} = \psi^p, \quad y \in \partial\mathbb{R}^n_+,\ -\infty < s < \infty, \tag{7.66}$$

$$0 \leqslant \psi(y,s) \leqslant 1, \qquad \psi_s(y,s) \geqslant 0 \quad y \in \mathbb{R}^n_+,\ -\infty < s < \infty, \tag{7.67}$$

$$\psi(0,0) \geqslant c_0 > 0. \tag{7.68}$$

By regularity for parabolic equations (e.g., a consequence of Theorem 3.2) we actually have $\psi \in C^\infty$. We shall now derive a contradiction by showing that there is no solution satisfying (7.65)–(7.68).

By monotone convergence theorem, the limit

$$\phi(y) = \lim_{s \to \infty} \psi(y,s) \tag{7.69}$$

is well-defined. We will now use the same method as in the proof of Theorem 6.1 (see Remark 6.1), using compactness and uniqueness of the limit (see (7.69)) to conclude that the above convergence (as $s \to \infty$, not just on subsequences) holds uniformly for any order of derivatives on any compact set. In particular,

$$\begin{aligned}\Delta_y \phi(y) &= \lim_{s\to\infty} \Delta_y \int_s^{s+1} \psi(y,\tau)d\tau \\ &= \lim_{s\to\infty} (\psi(y,s+1) - \psi(y,s)) \\ &= 0.\end{aligned}$$

Thus ϕ satisfies

$$\begin{aligned}&-\Delta_y \phi = 0, \quad y \in \mathbb{R}^n_+, \\ &\frac{\partial \phi}{\partial n_y} = \phi^p, \quad y \in \partial\mathbb{R}^n_+, \\ &0 \leqslant \phi(y) \leqslant 1, \quad y \in \mathbb{R}^n_+, \\ &\phi(0) > 0,\end{aligned}$$

which is a contradiction to Theorem 6.11(i). □

Remark 7.14. The above theorem is also valid for $p = n/(n-2)$. See [73].

Remark 7.15. The assumption that $\partial\Omega \in C^{1+\alpha}$ ensures that the scaled domain converges to a half-space. If we allow Lipschitz domains, then the scaled domain may converge to an infinite cone. In this case the result is valid for $1 < p < \dfrac{n-1}{n-2}$ (cf. [68]).

If we drop the monotonicity assumption in t, we have the following theorem.

Theorem 7.13. *Consider* (7.54)–(7.56)*, where we assume* (7.57)*,* (7.58) *and*

$$\lim_{\eta\to\infty} \frac{f(\eta)}{\eta^p} = 1, \quad 1 < p \leqslant 1 + \frac{1}{n}. \tag{7.70}$$

Then

$$\max_{x\in\overline{\Omega}} u(x,t) \leqslant \frac{C}{(T-t)^{1/[2(p-1)]}}, \tag{7.71}$$

where T is the blow-up time.

Proof. Note that (7.60) was used to prove that the solution of (7.65)–(7.68) is monotone in s in the proof of Theorem 7.12. In the case $1 < p \leqslant 1 + \frac{1}{n}$, (7.65)–(7.68) does not have a global solution by Theorem 5.6(ii). □

Remark 7.16. In the one-space-dimensional case with boundary flux u^p, with assumptions up to third order derivatives on the initial datum, the blow-up rate was established by Fila–Quittner [38].

Remark 7.17. Some of the methods here can also be used to study the blow-up rates for the equations with principle parts such as $u_t - \Delta u^{1+\sigma}$ or $u_t - \nabla(|\nabla u|^\sigma \nabla u^m)$, as well as systems such as (5.39) and (5.40). The blow-up rate estimate Theorem 7.13 uses the nonexistence part of the Fujita's critical exponents. The result was extended to system by Chlebik–Fila [24]. There are a flurry of works in blow-up properties, blow-up rate, etc., and we refer the readers to [13, 14, 22, 35, 40, 41, 44, 45, 92, 95, 98, 122, 126, 136–141, 145–147] and the references therein.

7.5 Exercises

7.1. Here is a "much simpler" proof for the blow-up rate upper bound for the system (7.1)–(7.3) than the Friedman–McLeod method:

Proof. Suppose that the solution of (7.1)–(7.3) blows up in finite time. We compare the solution of (7.1)–(7.3) with the solution of the ODE:

$$U'(t) = f(U(t)), \qquad U(0) = \max_{x\in\overline{\Omega}} u_0(x).$$

By the maximum principle, $u(x,t) \leqslant U(t)$, and therefore $\max_{x\in\overline{\Omega}} u(x,t) \leqslant U(t) \leqslant G(T-t)$, where T is the blow-up time and G is the inverse function of $\int_u^\infty \frac{d\eta}{f(\eta)}$, i.e., G satisfies $G\Big(\int_u^\infty \frac{d\eta}{f(\eta)}\Big) = u$. This proof does not even require the monotonicity of u with respect to t. This proof is *incorrect*. Indicate the error.

7.2. Can Corollaries 7.10, 7.11 be extended to the case $f(u) = u\ \log^p(u+1)$ for some $p > 1$?

7.3. Prove the following theorem:

Theorem 7.14. *Let Ω be a bounded domain with $\partial\Omega \in C^2$. Assume that f satisfies (7.10) and (7.19), and*

$$\int_c^\infty \frac{\eta d\eta}{f^2(\eta)} < \infty \quad \textit{for some } c > 0. \tag{7.72}$$

Prove that all nontrivial nonnegative solution to the system (7.5)–(7.7) must blow up in finite time.

(Hint: This is an exercise for scaling argument. (1) You may assume without loss of generality $u_t \geqslant 0$. Why? (2) If the solution is global in time, establish that $\lim_{t\to\infty} \max_{x\in\overline{\Omega}} u(x,t) = +\infty$.)

7.4. Prove (7.30) of Lemma 7.8.

7.5. Let Ω is a bounded smooth domain. Consider the problem

$$u_t = a_{ij}(x)u_{x_i x_j} + b_i(x)u_{x_i} + u^q, \quad x \in \Omega,\ t > 0,$$
$$\frac{\partial u}{\partial n} = u^p, \quad x \in \partial\Omega,\ t > 0,$$
$$u(x,0) = u_0(x) \geqslant 0,$$

where

$$a_{ij},\ b_i \in C^\alpha(\overline{\Omega}), \qquad \lambda I \leqslant (a_{ij}) \leqslant \Lambda, \quad \Lambda > \lambda > 0,$$
$$1 < p < (q+1)/2, \quad 1 < p \leqslant 1 + \tfrac{1}{n}.$$

Establish the blow-up rate estimate.

Chapter 8
Asymptotically Self-Similar Blow-Up Solutions

The similarity variables can be used to study the asymptotic behavior (see Barenblatt [9]). Giga–Kohn [64, 65] gave a finer description of the blow-up behavior of the equation

$$u_t - \Delta u = |u|^{p-1}u, \quad x \in \Omega, \;\; 0 < t < T, \tag{8.1}$$

$$u = 0, \quad x \in \partial\Omega, \;\; 0 < t < T, \tag{8.2}$$

$$u(x,0) = u_0(x). \tag{8.3}$$

Their approach also uses the similarity variables, together with the Pohozaev identity; see Theorem 8.5. In the one-space-dimensional case, the nonexistence result (and hence Theorem 8.5) was given earlier by Ad'jutov–Lepin [2] without using any Pohozaev-type inequalities.

For the problem

$$u_t - \Delta u = |u|^{p-1}u, \quad x \in \mathbb{R}^n, \;\; 0 < t < T, \tag{8.4}$$

we may look for special blow-up solution of the form

$$u(x,t) = (T-t)^{-1/(p-1)}w(y), \quad y = \frac{x}{\sqrt{T-t}}. \tag{8.5}$$

Such a solution is called a *backward self similar* blow-up solution. A simple computation shows that w is a solution of the PDE

$$-\Delta w + \frac{1}{2}y \cdot \nabla_y w + \frac{1}{p-1}w = |w|^{p-1}w, \qquad y \in \mathbb{R}^n. \tag{8.6}$$

Sometimes it is more convenient to rewrite the above equation in divergence form

$$-\nabla_y\Big(e^{-|y|^2/4}\nabla_y w\Big) = e^{-|y|^2/4}\Big(|w|^{p-1}w - \frac{1}{p-1}w\Big), \quad y \in \mathbb{R}^n. \tag{8.7}$$

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There are three constant solutions, given by

$$w \equiv 0, \qquad w = \pm\Big(\frac{1}{p-1}\Big)^{1/(p-1)}. \tag{8.8}$$

In the case $1 < p \leqslant \dfrac{n+2}{n-2}$, these are the only solutions as shown in the next section.

8.1 Pohozaev Identity

We assume that we can always integrate by parts over $\mathbb{R}^n$ in the computations. This is not a heavy restriction since there is a factor $e^{-|y|^2/4}$ in the equation. Multiplying (8.7) with w, and integrating over $\mathbb{R}^n$, we obtain

$$\int_{\mathbb{R}^n} e^{-|y|^2/4}|\nabla_y w|^2 dy = \int_{\mathbb{R}^n} e^{-|y|^2/4}\Big(|w|^{p+1} - \frac{1}{p-1}|w|^2\Big)dy. \tag{8.9}$$

Similarly, multiplying (8.7) with $|y|^2 w$, and integrating over $\mathbb{R}^n$, we get

$$\begin{aligned}&\int_{\mathbb{R}^n} e^{-|y|^2/4}|y|^2|\nabla_y w|^2 dy\\ &\quad = \int_{\mathbb{R}^n} e^{-|y|^2/4}|y|^2|w|^{p+1}dy + \int_{\mathbb{R}^n} e^{-|y|^2/4}\Big\{n|w|^2 - \frac{p+1}{2(p-1)}|y|^2|w|^2\Big\}dy.\end{aligned} \tag{8.10}$$

Finally, multiplying (8.7) with $y\cdot\nabla_y w$, and integrating over $\mathbb{R}^n$, we derive

$$\begin{aligned}&\int_{\mathbb{R}^n} e^{-|y|^2/4}\nabla_y w\cdot\nabla_y\Big(y\nabla_y w\Big)dy\\ &\quad = \int_{\mathbb{R}^n} ye^{-|y|^2/4}\cdot\nabla_y\Big(\frac{1}{p+1}|w|^{p+1} - \frac{1}{2(p-1)}\cdot|w|^2\Big)dy\\ &\quad = \int_{\mathbb{R}^n} \Big(\frac{|y|^2}{2} - n\Big)e^{-|y|^2/4}\cdot\Big(\frac{1}{p+1}|w|^{p+1} - \frac{1}{2(p-1)}\cdot|w|^2\Big)dy.\end{aligned} \tag{8.11}$$

Since

$$\nabla_y w\cdot\nabla_y\Big(y\nabla_y w\Big) = |\nabla_y w|^2 + \frac{1}{2}y\cdot\nabla_y\Big(|\nabla_y w|^2\Big),$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-|y|^2/4}\nabla_y w\cdot\nabla_y\Big(y\nabla_y w\Big)dy\\
&\quad=\int_{\mathbb{R}^n} e^{-|y|^2/4}|\nabla_y w|^2dy+\int_{\mathbb{R}^n} e^{-|y|^2/4}\ \frac{1}{2}\ y\cdot\nabla_y\Big(|\nabla_y w|^2\Big)dy \qquad (8.12)\\
&\quad=\int_{\mathbb{R}^n} e^{-|y|^2/4}|\nabla_y w|^2dy+\int_{\mathbb{R}^n} \frac{1}{2}\Big(\frac{|y|^2}{2}-n\Big)e^{-|y|^2/4}|\nabla_y w|^2dy.
\end{aligned}$$

Substituting (8.12) into (8.11), we derive

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-y^2/4}\Big(\frac{|y|^2}{4}+\frac{2-n}{2}\Big)|\nabla_y w|^2dy\\
&\quad=\int_{\mathbb{R}^n}\Big(\frac{|y|^2}{2}-n\Big)e^{-y^2/4}\cdot\Big(\frac{1}{p+1}|w|^{p+1}-\frac{1}{2(p-1)}\cdot|w|^2\Big)dy.
\end{aligned} \qquad (8.13)$$

The combination $2n\times(8.9)-(8.10)+2(p+1)\times(8.13)$ gives

$$\int_{\mathbb{R}^n}\Big((2-n)p+(n+2)+\frac{p-1}{2}|y|^2\Big)e^{-y^2/4}|\nabla_y w|^2dy=0.$$

To make the process rigorous, we need conditions near ∞ so that the above integrations can be carried out rigorously. Note that we have an exponentially decaying factor $e^{-|y|^2/4}$, the condition of w near ∞ is rather mild. For example, if $|w|$ grows not too fast near ∞, then all derivatives also grow not too fast near ∞ (left as Exercise 8.1). In this case we have

Theorem 8.1 (Pohozaev identity). *If $w\in C^2(\mathbb{R}^n)$ satisfies (8.6) and that $|w(y)|\leqslant C\exp(\varepsilon|y|^2)$ for some $0<\varepsilon\ll 1$, then*

$$\int_{\mathbb{R}^n}\Big((2-n)p+(n+2)+\frac{p-1}{2}|y|^2\Big)e^{-y^2/4}|\nabla_y w|^2dy=0. \qquad (8.14)$$

Corollary 8.2. *Assume that $1<p\leqslant(n+2)/(n-2)$, then a solution of (8.6) growing no faster than $\exp(\varepsilon|y|^2)$ $(\varepsilon\ll 1)$ at ∞ must be a constant.*

8.2 Asymptotically Backward Self Similar Blow-Up Solutions

The proof in this section is from Giga–Kohn [64, 65]. Consider (8.1)–(8.3). Using a translation if necessary, we assume that $0\in\Omega$ is a blow-up point.

We assume T is the blow-up time. After making a change of variables,

$$u(x,t)=(T-t)^{-1/(p-1)}w(y,s),\quad y=\frac{x}{\sqrt{T-t}},\ s=\log\frac{1}{T-t}, \qquad (8.15)$$

equation (8.1)–(8.3) is transformed to

$$w_s - \Delta_y w + \frac{1}{2} y \cdot \nabla_y w + \frac{1}{p-1} w = |w|^{p-1} w \quad \text{in } W, \tag{8.16}$$

$$w = 0 \quad \text{on} \bigcup_{s_0 \leqslant s < \infty} \partial\Omega(s), \tag{8.17}$$

$$w(y, s_0) = w_0(y) := T^{1/(p-1)} u_0(\sqrt{T}\, y), \tag{8.18}$$

where

$$W = \{(y,s);\ s_0 < s < \infty,\ e^{-s/2} y \in \Omega\},$$
$$\Omega(s) = \{y;\ (y,s) \in W\}, \quad s_0 = -\log T.$$

Equation (8.16) can be written in the divergence form:

$$e^{-y^2/4} w_s - \nabla_y \left(e^{-y^2/4} \nabla_y w \right) = e^{-y^2/4} \left(|w|^{p-1} w - \frac{1}{p-1} w \right) \quad \text{in } W. \tag{8.19}$$

Note that the study of the blow-up behavior as $t \nearrow T$ is now reduced to studying the behavior of w as $s \to \infty$. If $w(y,s)$ converges to a stationary solution, then w_s should converge to 0 as $s \to \infty$ in a certain sense. The idea of Giga–Kohn [64, 65] is to use a certain *energy* estimate to derive a bound for $\iint_W \rho(y) |w_s(y,s)|^2 dyds$ for some positive function $\rho(y)$, which then implies that w_s must converge to 0 as $s \to \infty$ in some weak sense. This is then used to derive that w must converge to a stationary solution on a subsequence $s = s_j \to \infty$. Such a subsequential limit is called a ω-limit. In the case $1 < p < (n+2)/(n-2)$, the Pohozaev identity implies that the subsequential limit can only be one of the three constants (uniqueness), therefore concluding the convergence as $s \to \infty$ in a similar way as in Remark 6.1.

We start with the definition of the "energy":

$$E(s) = \int_{\Omega(s)} \left(\frac{|\nabla w|^2}{2} + \frac{|w|^2}{2(p-1)} - \frac{|w|^{p+1}}{p+1} \right) \rho(y) dy, \quad \rho(y) = e^{-|y|^2/4}. \tag{8.20}$$

We begin with a lemma (see [97]).

Lemma 8.3. *If $f(y,s) : W \mapsto R$ is a smooth function, then*

$$\frac{d}{ds} \int_{\Omega(s)} f(y,s) dy = \int_{\Omega(s)} f_s(y,s) dy + \frac{1}{2} \int_{\partial\Omega(s)} f(y,s)(y \cdot \mathbf{n}) dS, \tag{8.21}$$

$$\begin{aligned} \frac{d}{ds} \int_{\partial\Omega(s)} f(y,s) dS = &\int_{\partial\Omega(s)} f_s(y,s) dS \\ &+ \frac{1}{2} \int_{\partial\Omega(s)} \left[(n-1) f(y,s) + \nabla f(y,s) \cdot y \right] dS, \end{aligned} \tag{8.22}$$

where $\mathbf{n}$ is the exterior normal vector of $\partial\Omega(s)$ and dS is the surface area element.

Proof. Since
$$\int_{\Omega(s)} f(y,s)dy = e^{ns/2}\int_{\Omega} f(xe^{s/2},s)dx,$$
we have
$$\begin{aligned}
&\frac{d}{ds}\int_{\Omega(s)} f(y,s)dy \\
&= \frac{n}{2}e^{ns/2}\int_{\Omega} f(xe^{s/2},s)dx \\
&\quad + e^{ns/2}\int_{\Omega}\Big(f_s(xe^{s/2},s) + \nabla_y f(xe^{s/2},s)\cdot \frac{1}{2}xe^{s/2}\Big)dx \\
&= \frac{n}{2}\int_{\Omega(s)} f(y,s)dy + \int_{\Omega(s)}\Big(f_s(y,s) + \nabla_y f(y,s)\cdot\frac{1}{2}y\Big)dy.
\end{aligned}$$

Thus (8.21) follows from integration by parts. The proof of (8.22) is left as an exercise. □

We now derive an "energy identity."

Lemma 8.4.
$$\int_{\Omega(s)} |w_s|^2 \rho dy = -\frac{d}{ds}E(s) - \frac{1}{4}\int_{\partial\Omega(s)} (y\cdot\mathbf{n})\Big|\frac{\partial w}{\partial n}\Big|^2 \rho dS. \tag{8.23}$$

Proof. Using (8.19) and formula (8.21), noting also that $w = 0$ on $\partial\Omega(s)$, we derive
$$\begin{aligned}
\frac{dE(s)}{ds} &= \int_{\Omega(s)}\Big(\nabla w\cdot\nabla w_s + \frac{1}{p-1}ww_s - |w|^{p-1}ww_s\Big)\rho dy \\
&\quad + \frac{1}{4}\int_{\partial\Omega(s)} |\nabla w|^2 (y\cdot\mathbf{n})dS \\
&= \int_{\Omega(s)} w_s\Big\{\Big(\frac{1}{p-1}w - |w|^{p-1}w\Big)\rho - \nabla(\rho\nabla w)\Big\}dy \\
&\quad + \int_{\partial\Omega(s)} w_s\frac{\partial w}{\partial n}\rho dS + \frac{1}{4}\int_{\partial\Omega(s)} |\nabla w|^2(y\cdot\mathbf{n})dS \\
&= -\int_{\Omega(s)} |w_s|^2\rho dy + \int_{\partial\Omega(s)} w_s\frac{\partial w}{\partial n}\rho dS + \frac{1}{4}\int_{\partial\Omega(s)} |\nabla w|^2(y\cdot\mathbf{n})dS.
\end{aligned}$$

Since $w(xe^{s/2},s) = 0$ for $x\in\partial\Omega$, we derive
$$\frac{1}{2}xe^{s/2}\cdot\nabla_y w(xe^{s/2},s) + w_s(xe^{s/2},s) = 0 \quad \text{for } x\in\partial\Omega,$$

i.e.,

$$\frac{1}{2}y \cdot \nabla_y w + w_s = 0 \quad \text{for } y \in \partial\Omega(s).$$

Since the tangential derivatives of w vanish, we obtain

$$\nabla_y w = \frac{\partial w}{\partial n}\, \mathbf{n},$$

so that

$$\int_{\partial\Omega(s)} w_s \frac{\partial w}{\partial n} \rho dS = -\frac{1}{2} \int_{\partial\Omega(s)} \left|\frac{\partial w}{\partial n}\right|^2 (y \cdot n) dS,$$

and the lemma follows. □

Theorem 8.5. *If* $0 \in \Omega$ *is a blow-up point,* Ω *is a bounded domain with* $\partial\Omega \in C^{2+\alpha}$, $1 < p \leqslant (n+2)/(n-2)$*, and*

$$|u(x,t)| \leqslant \frac{C}{(T-t)^{1/(p-1)}} \quad \text{in } \Omega \times (0,T), \tag{8.24}$$

then for any $K > 1$*,*

$$(T-t)^{1/(p-1)} u(y\sqrt{T-t}, t) \to c \quad \text{uniformly for } |y| \leqslant K \text{ as } t \nearrow T, \tag{8.25}$$

where c *is either* 0, $-(p-1)^{-1/(p-1)}$ *or* $(p-1)^{-1/(p-1)}$.

Proof. Under the assumptions it is not difficult to prove (left as an exercise)

$$\left|\frac{\partial}{\partial x_j} u(x,t)\right| \leqslant \frac{C}{(T-t)^{(p+1)/2(p-1)}}, \tag{8.26}$$

$$\left|\frac{\partial}{\partial t} u(x,t)\right|, \quad \left|\frac{\partial^2}{\partial x_i \partial x_j} u(x,t)\right| \leqslant \frac{C}{(T-t)^{p/(p-1)}}, \tag{8.27}$$

which implies that the function $w(y,s)$ satisfies

$$|w(y,s)|, \quad |w_{y_j}(y,s)|, \quad |w_{y_i y_j}(y,s)| \leqslant C \quad \text{for } (y,s) \in W. \tag{8.28}$$

It follows that

$$\sup_{s>s_0} |E(s)| < \infty,$$

and hence by Lemma 8.4

$$\iint_W |w_s|^2 \rho dy < \infty. \tag{8.29}$$

For any $K \geqslant 1$ and $S \gg 1$, by the Schauder estimate (Theorem 3.2) we have $C^{2+\alpha,1+\alpha}$ estimate for w on any set $\{|y| \leqslant K\} \times [S, S+1]$. The constant in this estimate is independent of S.

We now follow the procedure outlined in Remark 6.1. For any $s_j \nearrow \infty$, take subsequence (still denoted by s_j) such that

$$w(y, s+s_j) \to \psi(y, s) \quad \text{in } C^{2,1}(\{|y| \leqslant K\} \times [0,1]) \ \text{ for any } K > 1. \tag{8.30}$$

It is clear that ψ satisfies (8.16) on $\mathbb{R}^n \times [0,1]$. Since

$$\int_0^1 \int_{\Omega(s+s_j)} |w_s(y, s+s_j)|^2 \rho dy ds \leqslant \int_{s_j}^{\infty} \int_{\Omega(s)} |w_s(y,s)|^2 \rho dy ds \to 0$$

as $s_j \to \infty$, we must have

$$\int_0^1 \int_{\mathbb{R}^n} |\psi_s(y,s)|^2 \rho dy ds = 0,$$

and therefore $\psi_s \equiv 0$ in $\mathbb{R}^n \times [0,1]$. By the Pohozaev's identity, we must have $\psi \equiv 0, -(p-1)^{-1/(p-1)}$ or $(p-1)^{-1/(p-1)}$.

We next prove that the subsequential limit of $w(y,s)$ can only be one of these constants. If this were not true, then we could take a β between the two constants. By the intermediate value theorem, there exist $s_j \nearrow \infty$ such that $w(0, s_j) = \beta$, following the above argument we obtain a further subsequential limit $\psi(y,s)$ such that $\psi(0,0) = \beta$, ψ satisfies (8.16), and $\psi_s \equiv 0$. This is a contradiction.

For any sequence s_j, there exists a convergent subsequential limit. This limit is independent of the choices of the sequence s_j (unique). Thus $w(y,s)$ converges as $s \nearrow \infty$ uniformly on $|y| \leqslant K$ (see Remark 6.1). □

Remark 8.1. Actually, the limit in the above theorem cannot be zero [66].

Remark 8.2. This energy estimate can actually be used to establish the blow-up rate estimates for subcritical p's without the assumption $u_t \geqslant 0$:

Theorem 8.6 (Giga–Kohn[65]). *If $1 < p < (n+2)/(n-2)$ and Ω is a smooth and convex domain, then for the nonnegative solution $u(x,t)$ of (8.1)–(8.3):*

$$u(x,t) \leqslant \frac{C}{(T-t)^{1/(p-1)}},$$

where T is the blow-up time.

Remark 8.3. If $L[w](s)$ is *bounded from below* and satisfies

$$\frac{d}{ds} L[w](s) = -\int_{\Omega(s)} \rho(y, w, w_y) |w_s|^2 dy \leqslant 0 \quad \text{ for } s \gg 1,$$

where $\rho > 0$, then $L[w]$ is called *a Liapunov function*. From (8.23) we can easily construct a Lyapunov function for the corresponding equation.

8.3 A Method for Studying Asymptotic Behavior

We summarize the ingredients used to prove convergence in the previous section:

1. We need compactness so that we can take a subsequential limit. This is usually accomplished by PDE estimates.
2. We need a Lyapunov function to show that $w_s(y, s + s_j)$ converges to zero in some weak sense.
3. If the corresponding stationary solutions are discrete, then the convergence is true for $s \to \infty$, not just on subsequences.

Without (3), different subsequences may converge to different stationary solutions and we obtain a continuum of stationary solutions. These solutions are called *ω-limits*.

The Lyapunov function constructed in the previous section contains a factor $\rho(y) = \exp(-|y|^2/4)$. This is possible because the principle part of our equation is linear.

8.4 Constructing Lyapunov Functions in One Space Dimension

In the one-space-dimensional case, there is a general method for constructing Lyapunov functions (see Galaktionov [50], Zelenyak [148]). Consider the problem

$$w_s = b_1(y, w, w_y)\{w_{yy} + b_2(y, w, w_y)\}. \tag{8.31}$$

Take a function $\Phi = \Phi(y, w, q)$. We look for Lyapunov function of the form

$$L[w](s) = \int_{R_1(s)}^{R_2(s)} \Phi(y, w(y, s), w_y(y, s))dy \tag{8.32}$$

Then

$$\begin{aligned}
\frac{d}{ds}L[w](s) &= \int_{R_1(s)}^{R_2(s)} (\Phi_w w_s + \Phi_q w_{ys})dy + J_1 \\
&= \int_{R_1(s)}^{R_2(s)} \Big(\Phi_w - \Phi_{qy} - \Phi_{qw} w_y - \Phi_{qq} w_{yy}\Big) w_s dy + J_2 + J_1 \\
&= -\int_{R_1(s)}^{R_2(s)} \frac{\Phi_{qq}}{b_1}|w_s|^2 dy \\
&\quad + \int_{R_1(s)}^{R_2(s)} \Big(\Phi_w - \Phi_{qy} - \Phi_{qw} w_y + \Phi_{qq} b_2\Big) w_s dy + J_2 + J_1,
\end{aligned}$$

where J_1, J_2 are boundary integrals, and we assume that

$$\int_{s_0}^{\infty}(J_1(s)+J_2(s))ds<\infty. \tag{8.33}$$

We look for a function Φ such that

$$\Phi_w(y,w,q)-\Phi_{qy}(y,w,q)-\Phi_{qw}(y,w,q)q+\Phi_{qq}(y,w,q)b_2(y,w,q)=0, \tag{8.34}$$

$$\Phi_{qq}(y,w,q)>0. \tag{8.35}$$

Then $L[w](s)-\int_{s_0}^{s}(J_1(\eta)+J_2(\eta))d\eta$ is a Lyapunov function.

Differentiating (8.34) in q, we deduce

$$-\Phi_{qqy}(y,w,q)-\Phi_{qqw}(y,w,q)q+\Big(\Phi_{qq}(y,w,q)b_2(y,w,q)\Big)_q=0$$

Let

$$\rho(y,w,q)=\Phi_{qq}(y,w,q),$$

then (8.34) becomes

$$-\rho_q b_2+\rho_y+q\rho_w=(b_2)_q\rho. \tag{8.36}$$

This is a first order PDE for ρ which can be solved by the characteristic method: the characteristics are given by

$$\frac{dq}{dy}=-b_2(y,w,q),\qquad \frac{dw}{dy}=q.$$

Solving these equations, we find that $w=\phi(y)$ satisfies the steady-state equation of (8.31):

$$\phi_{yy}+b_2(y,\phi,\phi_y)=0. \tag{8.37}$$

We denote by $\phi(y_0,y,w_0,q_0)$ the solution of (8.37) satisfying the boundary condition

$$\phi\Big|_{y=y_0}=w_0,\qquad \phi_y\Big|_{y=y_0}=q_0. \tag{8.38}$$

Equation (8.37) can be solved for either $y<y_0$, or $y>y_0$. Rewriting (8.36) as

$$\begin{aligned}&\frac{d}{dy}\rho\Big(y,\phi(y_0,y,w_0,q_0),\phi_y(y_0,y,w_0,q_0)\Big)\\&\quad=(b_2)_q\Big(y,\phi(y_0,y,w_0,q_0),\phi_y(y_0,y,w_0,q_0)\Big)\\&\qquad\times\rho\Big(y,\phi(y_0,y,w_0,q_0),\phi_y(y_0,y,w_0,q_0)\Big),\end{aligned}$$

and integrating over $[0, y_0]$, we then have

$$\rho(y_0, w_0, q_0) = G\Big(\phi(y_0, 0, w_0, q_0), \phi_y(y_0, 0, w_0, q_0)\Big) \times \exp \int_0^{y_0} (b_2)_q \Big(\eta, \phi(y_0, \eta, w_0, q_0), \phi_y(y_0, \eta, w_0, q_0)\Big) d\eta, \tag{8.39}$$

where $G(w, q)$ is an arbitrary smooth function to be determined later on. Replacing (y_0, w_0, q_0) by (y, w, q) we obtain

$$\rho(y, w, q) = G\Big(\phi(y, 0, w, q), \phi_\eta(y, \eta, w, q)\Big|_{\eta=0}\Big) \times \exp \int_0^{y} (b_2)_q \Big(\eta, \phi(y, \eta, w, q), \phi_\eta(y, \eta, w, q)\Big) d\eta. \tag{8.40}$$

We now integrate twice in q to obtain Φ:

$$\Phi(y, w, q) = \int_0^q (q - \tau)\rho(y, w, \tau)d\tau + z_1(y, w) + qz_2(y, w). \tag{8.41}$$

If we let $\Phi_q(y, w, 0) = 0$, then $z_2 \equiv 0$. The function $z_1(y, w)$ can be determined by setting $q = 0$ in (8.34):

$$(z_1)_w(y, w) + \rho(y, w, 0)b_2(y, w, 0) = 0,$$

i.e.,

$$z_1(y, w) = -\int_0^w \rho(y, \eta, 0)b_2(y, \eta, 0)d\eta + g(y).$$

We shall take $g \equiv 0$.

We summarize the *procedure* for obtaining a Lyapunov function:

First solve the backward ODE (8.37) and (8.38). Then define ρ by (8.40) with the function G still to be determined. The form of G depends on the problem being considered. Finally, define

$$\Phi(y, w, q) = \int_0^q (q - \tau)\rho(y, w, \tau)d\tau + z_1(y, w), \tag{8.42}$$

where

$$z_1(y, w) = -\int_0^w \rho(y, \eta, 0)b_2(y, \eta, 0)d\eta. \tag{8.43}$$

We emphasize that this procedure is *formal*. Once the Lyapunov function is obtained, one needs to verify the various properties (such as integrability, bounds, etc.) needed for the Lyapunov function.

Remark 8.4. For (8.1) in the one-space-dimensional case with a single blow-up point, it is possible to modify the argument in this section to show that the solution in the similarity variable is stable. Although it is possible to have multiple blow-up points (see Remark 9.2), they are not stable.

8.5 Exercises

8.1. Let $w \in C^2(\mathbb{R}^n)$ be a solution of (8.6) and $|w| \leqslant C\exp(\varepsilon|y|^2)$ for some $0 < \varepsilon \ll 1$. Prove that there exists $C > 0$ such that,

$$\sum_{|\alpha|\leqslant 2} |D^\alpha w| \leqslant C\exp\left(\frac{1}{8}|y|^2\right) \quad \text{for } y \in \mathbb{R}^n.$$

8.2. Prove (8.22) of Lemma 8.3.

8.3. Prove (8.26) and (8.27).

8.4. Assume that the solution in the system (8.16) and (8.17) is radially symmetric. Use the method in Sect. 8.4 to construct a Lyapunov function. Rigorously establish your estimates and compare your Lyapunov function with (8.20).

Chapter 9
One Space Variable Case

The one-space-dimensional case as well as the radially symmetric case (which is essentially one-space-dimensional) is very special. In the one-space-dimensional case, a continuous curve in the x–t plane starting in the left half of the plane $\{(x,t);\ x<0\}$ cannot end up at the right half of the plane $\{(x,t);\ x>0\}$ without crossing the t-axis $\{x=0\}$. This situation *cannot* be extended to higher space dimensional case. The analysis of the sign of u_x can be used to study the blow-up set. For the equation

$$u_t = u_{xx} + f(u)$$

with Dirichlet boundary conditions, the single point blow-up was established by Weissler [135] for $f(u) = u^p$ and for certain special initial datum. Friedman–McLeod [42] showed that if the initial datum satisfies $u_0''(x) + f(u_0(x)) \geqslant 0$, the zeroth order compatibility condition, and changes the monotonicity only once (*one-peak*), then single point blow-up occurs. The two-peak situation was studied by Caffarelli–Friedman [17], and the general situation was studied in Chen–Matano [19].

Remark 9.1. It is worth emphasizing that the method in this chapter is strictly one-space-dimensional (or radially symmetric). One dimensional solutions may be used to study higher dimensional problems. Unfortunately, this method itself cannot be extended to higher dimensional case (You might ask the Monkey King for help). This is one of the many reasons that there are more results in the one-space-dimensional case.

9.1 Sturm Zero Number

For the solution of $u_t = a(x)u_{xx} + b(x)u_x + c(x)u$ with appropriate boundary conditions, the number of sign changes was studied as early as 1836 by Strum [128]. For this parabolic equation with either Dirichlet or Neumann boundary conditions on a bounded interval, the oscillation of the solution $u(\cdot,t)$ for any time $t>0$ will not be bigger than the initial datum. The number of sign changes is referred to as the

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Sturm zero number, and also known as the Lap number (see Matano [101]). When $a(x) = 1$, $b(x) = 0$, and f is a smooth function satisfying some growth conditions, Chen–Matano [19] showed that the blow-up set consists of finitely many points; the proof employs the idea of studying the zero set of a linear parabolic PDE from Angenent [4].

In this section we shall only use a special case to illustrate the method.

$$u_t = u_{xx} + u^p, \quad (x,t) \in \Omega_T = (0,1) \times (0,T), \quad 1 < p < \infty, \tag{9.1}$$

$$u(0,t) = u(1,t) = 0, \quad t > 0, \tag{9.2}$$

$$u(x,0) = u_0(x) \geqslant 0, \quad u_0 \in C^2[0,1], \quad u_0(0) = u_0(1) = 0. \tag{9.3}$$

Definition 9.1. For a function $f(x)$ defined on $[0,1]$, the Sturm zero number (or the Lap number) $L[f;[0,1]]$, counts the number of monotone pieces.

If $f \in C^1[0,1]$, we define $N[f,[a,b]]$ to be the indicator function of monotonicity change of f between the two points $x = a$ and $x = b$, i.e.,

$$N[f,[a,b]] = 1 \quad \text{if } f'(a)f'(b) < 0, \quad N[f,[a,b]] = 0 \quad \text{otherwise}. \tag{9.4}$$

Then

$$L[f;[0,1]] = 1 + \sup\left\{\sum_{j=1}^{n} N[f,[x_{j-1},x_j]];\ \ 0 = x_0 < x_1 < \cdots < x_n = 1\right\}.$$

The following theorem is the First Strum Theorem on sign changes stated for the system (9.1)–(9.3).

Theorem 9.1. *Let u be a classical solution of (9.1)–(9.3). Then $L[u(\cdot,t);[0,1]]$ is monotone nonincreasing in t.*

Proof. We want to establish

$$L[u(\cdot,t_0);[0,1]] \leqslant L[u_0;[0,1]]. \tag{9.5}$$

By Hopf's Lemma (Theorem 3.6)

$$u_x(0,t) > 0, \quad u_x(1,t) < 0, \quad 0 < t < T. \tag{9.6}$$

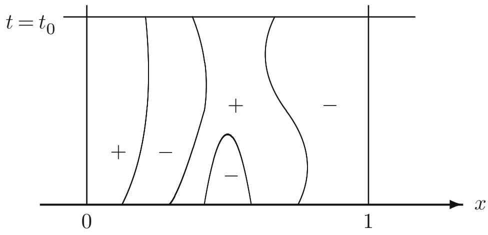
Differentiating the equation in x, we obtain

$$(u_x)_t = (u_x)_{xx} + pu^{p-1}u_x, \quad x \in (0,1),\ t > 0. \tag{9.7}$$

For any connected (open) component P of the set $\{(x,t) \in \Omega_{t_0};\ u_x(x,t) > 0\}$ ($\Omega_{t_0} = (0,1) \times (0,t_0)$), we have

$$u_x(x,t) = 0 \quad \text{on } \partial P \cap \Omega_{t_0}. \tag{9.8}$$

Fig. 9.1 Sign of u_x



If $u_x \equiv 0$ on ∂P, then $u_x \equiv 0$ on $\overline{P}$ by the maximum principle, which is a contradiction (Fig. 9.1). Therefore,

$$\Gamma_1 := \partial P \cap \{(x,t) \in \overline{\Omega}_{t_0};\ u_x(x,t) > 0\} \neq \emptyset. \tag{9.9}$$

In view of (9.8) and (9.9), Γ_1 must intersect the parabolic boundary $\partial_p \Omega_{t_0} = (\{x = 0, 1\} \times [0, t_0]) \cup ([0,1] \times \{t = 0\})$.

In view of (9.6), Γ_1 cannot intersect $\{x = 1\} \times [0, t_0]$.

If Γ_1 does not intersect $[0,1] \times \{t = 0\}$, then we can apply the maximum principle on the domain bounded by $\partial P \cap \Omega_{t_0}$ and the t-axis and using (9.6) to conclude that $u_x > 0$ there. Since this domain is connected, we conclude that this domain must be P itself. On the other hand, we can also apply the maximum principle on P using the boundary condition $(u_x)_x(0,t) = 0$ instead of (9.6) to conclude $u_x \equiv 0$ in P, which is a contradiction.

We thus proved

$$\Gamma_1 \cap ([0,1] \times \{t = 0\}) \neq \emptyset.$$

We can follow the same procedure to prove that a connected component of $\{(x,t) \in \overline{\Omega}_{t_0};\ u_x(x,t) < 0\}$ must contain an interval on the x-axis.

Thus, each connected component of either $\{(x,t) \in \overline{\Omega}_{t_0};\ u_x(x,t) > 0\}$ or $\{(x,t) \in \overline{\Omega}_{t_0};\ u_x(x,t) < 0\}$ must contain an interval on the x-axis.

In particular, if $0 < x_0 < x_1 < \cdots < x_m < 1$ such that $u_x(x_j, t_0) \cdot u_x(x_{j+1}, t_0) < 0$ $(j = 0, 1, \cdots, m-1)$, we can find $0 < y_0 < y_1 < \cdots < y_m < 1$ so that $(y_j, 0)$ can be connected to (x_j, t_0) by a curve on which u_x has the same sign, and therefore (9.5) follows. □

9.2 Finite-Points Blow-Up

Theorem 9.2. *Consider (9.1)–(9.3). If $u_0(x)$ has k local maxima in $[0,1]$, then the blow-up set consists of no more than k points in $(0,1)$.*

Proof. **Step 1.** By Theorem 9.1, we may assume that, for $0 < T - t \ll 1$,

$$(-1)^j u_x(x,t) > 0 \quad \text{for } \xi_j(t) < x < \xi_{j+1}(t), \quad j = 0, 1, \cdots, 2m-1, \tag{9.10}$$

where $m \leqslant k$, $\xi_0(t) \equiv 0$ and $\xi_{2m}(t) \equiv 1$. In this step, we want to show that the limits

$$\lim_{t \nearrow T} \xi_j(t), \quad j = 1, 2, \cdots, 2m-1$$

exist. Suppose this were not true, then for some $1 \leqslant j \leqslant 2m-1$,

$$\beta_1 = \liminf_{t \nearrow T} \xi_j(t) < \limsup_{t \nearrow T} \xi_j(t) = \beta_2. \tag{9.11}$$

We take $\alpha \in (\beta_1, \beta_2), \alpha \neq 1/2$, and consider the domain

$$Q := \{(x,t);\ \ \alpha < x < \min(1, 2\alpha),\ T - \varepsilon < t < T\}, \qquad (0 < \varepsilon \ll 1).$$

The function $\psi(x,t) = u(x,t) - u(2\alpha - x, t)$ satisfies

$$\psi_t - \Delta\psi = c(x,t)\psi \quad \text{in } Q, \tag{9.12}$$

$$\psi(\alpha, t) = 0, \quad t > 0, \tag{9.13}$$

$$\psi(1,t) < 0,\ t > 0 \quad \text{if } \alpha < \frac{1}{2}; \quad \psi(2\alpha, t) > 0,\ t > 0 \quad \text{if } \alpha > \frac{1}{2}. \tag{9.14}$$

We assume for definiteness that

$$\alpha < \frac{1}{2}.$$

By (9.11), $\psi_x(\alpha,t) = 2u_x(\alpha,t)$ must change sign infinitely many times as $t \nearrow T$. At the point where $\psi_x(\alpha,t) > 0$, there exists $x^+(t) > 0$ such that $\psi(x,t) > 0$ for $0 < x \leqslant x^+(t)$. At the point where $\psi_x(\alpha,t) < 0$, there exists $x^-(t) > 0$ such that $\psi(x,t) < 0$ for $0 < x \leqslant x^-(t)$.

Take $0 < T - t_0 \ll 1$ such that $\psi_x(\alpha, t_0) > 0$ and let P be a connected component of the set $\{(x,t) \in Q;\ \ \psi(x,t) > 0\}$ containing $(x^+(t_0), t_0)$. If $\psi = 0$ on ∂P, then $\psi \equiv 0$ in P, which is a contradiction. Thus

$$\partial P \cap \{(x,t);\ \psi(x,t) > 0\} \neq \emptyset. \tag{9.15}$$

Since $\psi(1,t) < 0$, ∂P cannot intersect $\{x = 1\}$. It is also clear that

$$\psi = 0 \quad \text{on } \partial P \cap Q.$$

Therefore, $\partial P \cap \{(x,t);\ \psi(x,t) > 0\}$ must intersect $[\alpha, 1] \times \{t = T - \varepsilon\}$, and there is a continuous curve $\Gamma_{t_0}^+$ in P connecting a point $(a^+(t_0), T-\varepsilon)$ $(\alpha < a^+(t_0) < 1)$ (on which $\psi(a^+(t_0), T - \varepsilon) > 0$) to (α, t_0).

Now take $T - \varepsilon < t_1 < t_0$ such that $\psi_x(\alpha, t_1) < 0$. A connected component of $\{(x,t) \in Q;\ \ \psi(x,t) < 0\}$ containing $(x^-(t_1), t_1)$ cannot intersect $\Gamma_{t_0}^+$, and therefore cannot have $\{x = 1\}$ as part of its boundary. Using a similar argument as above, there exists a $a^-(t_1)$, $\psi(a^-(t_1), T - \varepsilon) < 0$ and a curve $\Gamma_{t_1}^-$ (on which $\psi < 0$)

connecting $(a^-(t_1), T-\varepsilon)$ to (α, t_1). By construction, we must have $a^-(t_1) < a^+(t_0)$.

Repeating this argument and letting $t_0 \nearrow T$, we find that ψ must change sign on $T-\varepsilon$ infinitely many times. Since $\psi(1, T-\varepsilon) < 0$, $\psi(\cdot, T-\varepsilon)$ must change sign infinitely many times near a point x^* where both $u(x^*, T-\varepsilon)$ and $u(2\alpha - x^*, T-\varepsilon)$ remains positive in a neighborhood. This is a contradiction to the fact that $\psi(x, T-\varepsilon)$ is analytic near $x = x^*$.

(We remark here that the analyticity in x can be obtained by using the Schauder interior estimates).

Step 2. We next finish the proof by showing that if

$$u_x(x,t) \geqslant 0 \quad \text{for } \alpha \leqslant x \leqslant \beta,\ t_0 < t < T, \tag{9.16}$$

then $x = \beta$ is the only possible blow-up point. Suppose that $x = \alpha_1 < \beta$ is also a blow-up point. Then by (9.16), for $\alpha_2 \in (\alpha_1, \beta)$,

$$\limsup_{t \nearrow T} u(\alpha_2, t) = \infty.$$

A. We first show that $u \to \infty$ uniformly on any compact subset of $(\alpha_2, \beta]$.

Take $t_j \nearrow T$ such that $M_j := u(\alpha_2, t_j) \to \infty$. Since $u(x, t_j)$ is monotone in x,

$$u(x, t_j) \geqslant M_j \quad \text{for } \alpha_2 < x < \beta. \tag{9.17}$$

Let $v(x)$ be the solution of

$$v''(x) + v^p(x) = 0, \quad v(0) = 1, \quad v'(0) = 0;$$

the solution $v(x)$ can be solved explicitly, given by

$$\int_{v(x)}^{1} \frac{d\eta}{\sqrt{1-\eta^{p+1}}} = \sqrt{\frac{2}{p+1}}\ |x|,$$

and thus there exists a unique $x_p^0 > 0$ such that

$$v(\pm x_p^0) = 0, \quad v(x) > 0 \quad \text{for } |x| < x_p^0. \tag{9.18}$$

Clearly, $w_j(x) = M_j\ v(M_j^{(p-1)/2} x)$ satisfies

$$w_j''(x) + w_j^p(x) = 0, \quad w_j(0) = M_j, \tag{9.19}$$

$$w_j\Big(\pm \frac{x_p^0}{M_j^{(p-1)/2}}\Big) = 0, \quad w_j(x) > 0 \quad \text{for } |x| < \frac{x_p^0}{M_j^{(p-1)/2}}. \tag{9.20}$$

Recalling (9.17) and the fact that u is nonnegative, we can apply the maximum principle to deduce

$$u(x,t) \geqslant w\Big(x - \frac{x_p^0}{M_j^{(p-1)/2}} - \alpha_2\Big) \quad \text{for } \alpha_2 < x < \alpha_2 + 2\frac{x_p^0}{M_j^{(p-1)/2}},\ t > t_j.$$

In particular,

$$u\Big(\frac{x_p^0}{M_j^{(p-1)/2}} + \alpha_2\Big) \geqslant M_j \quad \text{for } t \geqslant t_j. \tag{9.21}$$

Thus, by monotonicity, for any $\alpha_3 \in (\alpha_2, \beta)$, we can take $M_j \gg 1$ such that $\dfrac{x_p^0}{M_j^{(p-1)/2}} + \alpha_2 < \alpha_3$, and so

$$u(\alpha_3, t) \geqslant M_j \quad \text{for } t \geqslant t_j. \tag{9.22}$$

Thus

$$\liminf_{t \nearrow T} u(\alpha_3, t) = \infty. \tag{9.23}$$

B. We take $\alpha_3 < \beta_1 < \beta$ and consider the function

$$J = u_x - c(x)u^q, \quad \alpha_3 < x < \beta_1,\ T - \varepsilon < t < T; \qquad 1 < q < p.$$

Then

$$\begin{aligned} &J_t - \Delta J - pu^{p-1}J - 2c'(x)qu^{q-1}J \\ &\quad \geqslant (p-q)c(x)u^{p+q-1} + 2c'(x)c(x)qu^{2q-1} + c''(x)u^q. \end{aligned}$$

We take $c(x) = \delta \sin\Big(\dfrac{\pi(x-\alpha_3)}{\beta_1 - \alpha_3}\Big)$, $0 < \delta \ll 1$. Then for $0 < \varepsilon \ll 1$, $u \gg 1$ uniformly for $\{\alpha_3 < x < \beta_1,\ \ t_1 < t < T\}$ by (9.23). Thus

$$J_t - \Delta J - pu^{p-1}J - 2c'(x)qu^{q-1}J \geqslant 0, \quad \alpha_3 < x < \beta_1,\ T - \varepsilon < t < T. \tag{9.24}$$

Notice that our choice of δ is independent of ε. By the maximum principle $u_x(x, T-\varepsilon) < -c_\varepsilon < 0$ for $\alpha_3 < x < \beta_1$. Thus if we take δ to be small enough, then we can apply the maximum principle to conclude $J > 0$ for $\alpha_3 < x < \beta_1,\ \ T - \varepsilon < t < T$.

Integrating the inequality

$$\frac{u_x}{u^q} \geqslant c(x), \quad \alpha_3 < x < \beta_1,\ \ T - \varepsilon < t < T \tag{9.25}$$

over the interval $\alpha_3 < x < \beta$, we obtain that $u(\alpha_3, t)$ is uniformly bounded, which is a contradiction.

Step 3. We can similarly show that if

$$u_x(x,t) \leqslant 0 \quad \text{for } \alpha \leqslant x \leqslant \beta,\ t_0 < t < T,$$

then $x = \alpha$ is the only possible blow-up point.

Step 4. Combining all the above steps, the blow-up point can only be the limit as $t \nearrow T$ (the limit exists, by Step 1) of local maxima of $u(\cdot,t)$. Thus we conclude the theorem. □

Remark 9.2. It is possible that the blow-up occurs at exactly finitely many given points [105].

Remark 9.3. The blow-up set is also studied for many other types of equations, e.g., [48].

Remark 9.4. It is natural to ask whether this isolated blow-up point occurs in the boundary source case. An example was constructed in [74] that isolated blow-up points can also occur in this case.

9.3 Intersection Comparison: An Example of Complete Blow-Up

The key to intersection comparison is that the number of zeros of a linear parabolic equation

$$u_t = a(x,t)u_{x_i x_j} + b(x,t)u_{x_i} + c(x,t)u, \quad (a(x,t) > 0)$$

cannot increase in time. Therefore if we compare two solutions of the nonlinear equation such as

$$u_t - \Delta u = f(u), \qquad u(x,t) = u(r,t), \qquad r = |x|;$$

the number of intersections cannot increase in time.

The intersection comparison is also essentially one-space-dimensional; this includes the radially symmetric case for several space dimensions.

The principle for intersection comparison is as follows:

1. Find a family of special solutions that has known properties; this is done usually through the analysis of the corresponding ODE.
2. Compare the solution with solutions found in (1). The difference can have at most finitely many sign changes; in many applications, the sign will change once under appropriate assumptions on the initial datum.
3. Use convexity, interface analysis, etc., to obtain the properties of the solution under investigation.

The intersection comparison can be found in the books by Galaktionov [51] and Samarskii–Galaktionov–Kurdyumov–Mikhailov [125]. It is used by Galaktionov and his co-authors to study various types of problems. The general ideas go back as early as Sturm [128] in 1836. Here we shall use the simplest model to illustrate how this method can be used to establish *complete* blow-up results. For the positive solution of the equation $u_t = \Delta u + u^p$, complete blow-up was established by Baras–Cohen [8] for all sub-critical p's; see also Galaktionov–Vàzquez [55–57] for porous medium equations with a nonlinear source. The proof here is from [56], and we shall only use the heat equation to illustrate the techniques here.

Consider the equation:

$$\begin{aligned} u_t &= A[u] + f(u) \quad \text{in } \Omega \times \{0 < t < T\}, \\ u &= 0 \qquad \text{on } \partial\Omega \times \{0 < t < T\}, \\ u(x,0) &= u_0(x) \geqslant 0 \quad \text{in } \Omega, \end{aligned} \tag{9.26}$$

where $f(u) > 0$ for $u > 0$

$$\int^{\infty} \frac{du}{f(u)} < \infty,$$

and A is an elliptic operator. We assume that A is either linear, or if A is nonlinear, the nonlinearity of A does not cause blow-up. In another words, we assume that the blow-up is solely attributed to the source $f(u)$. Some examples include $A[u] = \Delta u$, and $A[u] = \Delta u^m$.

If $f(u)$ in (9.26) is replaced by $f_M(u) := \max\Big(M, f(u)\Big)$, its corresponding solution is denoted by u_M. Since f_M is bounded, it is clear that u_M is well defined for all $0 < t < \infty$.

We assume that

1. For $t < T$, $u_M \to u$ as $M \to \infty$, so that u is a solution of (9.26);
2. T is a blow-up time for u, i.e., $\limsup_{t \nearrow T} \sup_{x\in\Omega} u(x,t) = +\infty$.

Note that even if u blows up at $t = T$, it is possible that $u(\cdot, T-)$ exists and is not identically $+\infty$. In this case an extension beyond $t > T$ may be possible, in this case we say that the blow-up is *incomplete*. A complete blow-up means that no extension is possible for $t > T$ after blow-up occurs, even in the weak sense.

Definition 9.2. If for every $t > T$,

$$\lim_{M\to\infty} u_M(x,t) = +\infty, \quad x \in \Omega,$$

we say that the blow-up is *complete.*

Consider now the radial solution of

$$\begin{aligned} &u_t = u_{rr} + \frac{n-1}{r} u_r + u^p, \quad u_r(0,t) = 0, \quad 0 < r < R,\ t > 0, \\ &u(R,t) = 0, \quad t > 0, \\ &u(r,0) = u_0(r) \geqslant 0. \end{aligned} \tag{9.27}$$

We begin with the ODE:

$$u_{rr} + \frac{n-1}{r}u_r + u^p = 0. \tag{9.28}$$

Let $U(r)$ be the solution of

$$U_{rr} + \frac{n-1}{r}U_r + U^p = 0, \quad U'(0) = 0 \tag{9.29}$$

such that

$$U(0) = 1. \tag{9.30}$$

Then $U(r)$ is well defined for $|r| \ll 1$. We next prove:

Lemma 9.3. *If* $1 < p < \dfrac{n+2}{n-2}$, $(1 < p < \infty$ *if* $n = 1, 2)$, *then there exists a* $r_0 > 0$ *such that*

$$U(r) > 0 \quad \textit{for } 0 \leqslant r < r_0, \qquad U(r_0) = 0, \tag{9.31}$$

$$U'(r) < 0 \quad \textit{for } 0 < r \leqslant r_0. \tag{9.32}$$

Proof. Let $[0, r_0)$ be the maximal interval on which $U(r)$ remains positive. Then by Theorem 6.10(i), $r_0 < \infty$ and hence $U(r_0) = 0$. (Using Theorem 6.10 to prove $r_0 < \infty$ looks like shooting a mosquito using a gun; one can prove $r_0 < \infty$ directly from the ODE, but the argument may not be as simple as one might expect).

Since

$$U'(r) = -\frac{1}{r^{n-1}}\int_0^r \tau^{n-1}U^p(\tau)d\tau,$$

the rest of the lemma follows. □

For notational convenience, we extend $U(r)$ to be zero for $r > r_0$.

Set

$$U(r, \lambda) = \lambda U(\lambda^{(p-1)/2}r), \tag{9.33}$$

then $U(r, \lambda)$ is a solution of (9.29) in the region $0 < r < \lambda^{-(p-1)/2}r_0$, and $U(r) = U(r, 1)$.

Define the *envelope* $\mathcal{L}(r)$ by

$$\mathcal{L}(r) = \sup_{\lambda > 0} U(r, \lambda). \tag{9.34}$$

Lemma 9.4. *If* $1 < p < \dfrac{n+2}{n-2}$, $(1 < p < \infty$ *if* $n = 1, 2)$, *then*

$$\mathcal{L}(r) \equiv c^* r^{-2/(p-1)}. \tag{9.35}$$

Proof. From (9.33), we derive

$$r^{2/(p-1)}U(r,\lambda) = \left(\lambda^{(p-1)/2}r\right)^{2/(p-1)} U(\lambda^{(p-1)/2}r).$$

It follows that

$$\sup_{\lambda>0} r^{2/(p-1)}U(r,\lambda) = \sup_{s>0} s^{2/(p-1)}U(s) = c^* > 0. \qquad \square$$

In this section, we shall discuss our problem under some additional simplifying assumptions. We assume that $r=0$ is a blow-up point, and

$$\limsup_{t \nearrow T} u(0,t) = +\infty. \tag{9.36}$$

Lemma 9.5. *Let* $u_0 \in C^1$, $1 < p < \dfrac{n+2}{n-2}$, $(1 < p < \infty$ *if* $n = 1,2)$, *and the assumption* (9.36) *be in force. Then there exists* $r_* > 0$ *such that*

$$\liminf_{t \nearrow T} u(r,t) > \mathcal{L}(r) \quad \textit{for } 0 < r < r_*; \tag{9.37}$$

we emphasize that this inequality is a strict inequality.

Proof. Take $\lambda > 0$ such that $\lambda^{-(p-1)/2}r_0 < R$. Let

$$w_\lambda(r,t) = u(r,t) - U(r,\lambda), \quad 0 < r < \lambda^{-(p-1)/2}r_0,\ t > 0. \tag{9.38}$$

Then clearly,

$$(w_\lambda)_t = (w_\lambda)_{rr} + \frac{n-1}{r}(w_\lambda)_r + c(x,t)w_\lambda, \qquad 0 < r < \lambda^{-(p-1)/2}r_0,\ \ 0 < t < T, \tag{9.39}$$

$$w_\lambda(\lambda^{-(p-1)/2}r_0, t) > 0, \quad 0 < t < T, \tag{9.40}$$

where $c(x,t)$ is a bounded function for $0 < t < T - \varepsilon$, for any fixed $\varepsilon > 0$.

For each $t > 0$, denote by $J[w_\lambda](t)$ the number of sign changes of $w_\lambda(\cdot,t)$ in the interval $r \in [0, \lambda^{-(p-1)/2}r_0]$, we leave it as an exercise to show that $J(t)$ is nonincreasing in time.

Since $u_0 \in C^1$, there exists $\Lambda \gg 1$ such that

$$J[w_\lambda](0) = 1 \quad \text{for } \lambda > \Lambda. \tag{9.41}$$

Therefore

$$J[w_\lambda](t) \leqslant 1 \quad \text{for } 0 < t < T,\ \lambda > \Lambda. \tag{9.42}$$

In view of (9.36), there exists $t_\lambda < T$ such that $w_\lambda(0, t_\lambda) > 0$, it follows that

$$J[w_\lambda](t_\lambda) = 0, \tag{9.43}$$

so that $t = t_\lambda < T$ such that

$$J[w_\lambda](t) = 0 \quad \text{for } t_\lambda < t < T, \tag{9.44}$$

in this case we must have (recalling (9.40))

$$w_\lambda(x,t) = u(x,t) - U(x,\lambda) > 0, \quad 0 < r < \lambda^{-(p-1)/2} r_0, \;\; t_\lambda < t < T. \tag{9.45}$$

Taking the supremum over $\lambda \in [\Lambda, \infty)$, and noticing that for each fixed $r \in (0, r_*)$, the supremum is reached, we conclude the lemma. □

This lemma is enough for establishing complete blow-up if we further restrict the range of p's.

Lemma 9.6. *Let* $u_0 \in C^1$, $1 < p \leqslant \dfrac{n}{n-2}$, $(1 < p < \infty$ *if* $n = 1, 2)$, *and the assumption* (9.36) *be in force. Then the blow-up is complete.*

Proof. Let

$$g_k(r) = \min(k, \mathcal{L}(r)) \quad \text{for } 0 < r < r_*, \qquad g_k(r) = 0 \quad \text{for } r \geqslant r_*. \tag{9.46}$$

In view of Lemma 9.5, it suffices to show that the solution $v_k(r,t)$ to the problem

$$v_t = \Delta v + g_k^p(r), \quad r < R, \;\; t > T, \tag{9.47}$$

$$v(r,T) = g_k(r), \quad r < R, \tag{9.48}$$

$$v(R,t) = 0, \quad t > T \tag{9.49}$$

satisfies, for any $\delta > 0$,

$$\lim_{k\to+\infty} v_k(r, T+\delta) = +\infty \quad \text{for } \delta > 0, \; 0 < r < R - \delta. \tag{9.50}$$

In fact, since $1 < p \leqslant \dfrac{n}{n-2}$, we have $-\dfrac{2p}{p-1} + (n-1) \leqslant -1$, and hence

$$\begin{aligned} \|g_k^p\|_{L^1(B_R(0))} &= \int_{B_R(0)} g_k^p(|x|)dx \\ &= n\omega_n \int_0^{r_*} g_k^p(r) r^{n-1} dr \nearrow +\infty \quad \text{as } k \to +\infty, \end{aligned}$$

(here $n\omega_n$ is the surface area of a unit n-dimensional sphere). It follows that

$$v_k(|x|,t) \geqslant \int_0^{t-T} \int_{B_R(0)} G(x,y,t,\tau) g_k^p(|y|) dy d\tau, \qquad t > T,$$

where $G(x,y,t,\tau)$ is the Green's function for the Dirichlet problem on $B_R(0)$. Since $G(x,y,t,\tau)$ is bounded from below by a positive constant c_δ on the set

$$\{(x,y,t,\tau);\ \ \delta/2 < t-\tau < \delta,\ R-|x| > \delta,\ R-|y| > \delta\}, \quad (\delta > 0),$$

we conclude, for $r < R-\delta$,

$$\begin{aligned} \liminf_{k\to\infty} v_k(r, T+\delta) &\geqslant \liminf_{k\to\infty} \int_0^{t-T} \int_{B_R(0)} G(x,y,t,\tau) g_k^p(|y|) dy d\tau \\ &\geqslant \frac{c_\delta \delta}{2} \liminf_{k\to\infty} \|g_k^p\|_{L^1(B_{r_*}(0))} = +\infty. \qquad \square \end{aligned}$$

In the case $p > \dfrac{n}{n-2}$, there is a singular stationary solution

$$S(r) = cr^{-2/(p-1)}, \qquad c = \Big[\frac{2}{p-1}\Big(n-2-\frac{2}{p-1}\Big)\Big]^{1/(p-1)}. \tag{9.51}$$

and

$$\int_0^R S(r) r^{n-1} dr < \infty.$$

In order to study the problem when p is in the range $\Big(\dfrac{n}{n-2}, \dfrac{n+2}{n-2}\Big)$, we consider an auxiliary problem:

$$\phi_t = \Delta\phi + \phi^p, \quad r < r_1, \ \ t > 0, \tag{9.52}$$

$$\phi(r_1,t) = 0, \quad t > 0, \tag{9.53}$$

$$\phi(r_1,0) = U(r), \quad r < r_1, \tag{9.54}$$

where we take $r_1 > r_0$, and r_0 defined by Lemma 9.3. Recall that $U(r)$ is defined by (9.29) and (9.30) and is extend to be 0 for $r \geqslant r_0$.

Lemma 9.7. *ϕ must blow up at a finite time, say $t = T_1$, and*

$$\lim_{t\nearrow T_1} \phi(0,t) = +\infty. \tag{9.55}$$

Proof. It is clear that U satisfies

$$U'' + U^p = 0 \quad \text{for } r < r_0, \quad \text{and for } r > r_0.$$

Since the corner at $r = r_0$ is *convex*, it is not difficult to prove that there exists $U_k \in C^2[0,\infty)$ $(k = 1, 2, 3, \cdots)$ such that

$$U_k'' + U_k^p \geqslant 0 \quad \text{for } r < \infty, \quad U_k \equiv U \quad \text{for } |r - r_0| \leqslant \frac{1}{k}.$$

By using an approximation using U_k as the initial datum and then taking the limit as $k \to \infty$, we derive

$$\phi_t \geqslant 0, \quad \phi_r < 0 \quad \text{for } 0 \leqslant r < r_1,\ t > 0.$$

Since the initial datum is not smooth and $\phi(\cdot, t)$ is smooth for $t > 0$, we must have $\phi_t \not\equiv 0$, and so

$$\phi_t > 0 \quad \text{for } 0 \leqslant r < r_1,\ t > 0. \tag{9.56}$$

By Hopf's lemma, $(\phi_t)_r(r_1, \varepsilon) < 0$ for $0 < \varepsilon \ll 1$. It follows that

$$\phi_t(r, \varepsilon) - \delta\phi^p(r, \varepsilon) \geqslant 0, \quad r \leqslant r_1, \quad \text{for } 0 < \delta \ll 1.$$

Therefore by the maximum principle, $\phi_t \geqslant \delta\phi^p$ in the domain and it must blow-up in finite time, which we denote by T_1. Since $\phi_r < 0$ and $\phi_t > 0$, (9.55) follows. □

Theorem 9.8. *Let* $u_0 \in C^1$, $1 < p < \dfrac{n+2}{n-2}$, $(1 < p < \infty$ *if* $n = 1, 2)$, *and the assumption* (9.36) *be in force. Then the blow-up is complete.*

Proof. The case $1 < p \leqslant n/(n-2)$ has been considered. We now consider the case

$$n \geqslant 3, \qquad \frac{n}{n-2} < p < \frac{n+2}{n-2}. \tag{9.57}$$

Denote by T the blow-up time for $u(r, t)$ and set

$$g_k(r) = \min_{0 \leqslant \xi \leqslant r} h_k(\xi), \quad h_k(r) = \min\Big(k,\ \liminf_{t \nearrow T} u(r, t)\Big). \tag{9.58}$$

Clearly,

$$\frac{d}{dr} g_k(r) \leqslant 0, \qquad 0 \leqslant g_k(r) \leqslant k.$$

It suffices to show that the solution $v_k(r, t)$ to the problem

$$v_t = \Delta v + v^p(r, t), \quad r < R, \ \ t > T, \tag{9.59}$$

$$v(r, T) = g_k(r), \quad r < R, \tag{9.60}$$

$$v(R, t) = 0, \quad t > T \tag{9.61}$$

satisfies, for any $\delta > 0$,

$$\lim_{k\to+\infty} v_k(r, T+\delta) = +\infty \quad \text{for } \delta > 0,\ 0 < r < R - \delta. \tag{9.62}$$

When p is in the range (9.57), it is clear that $\phi^\lambda(r,t) = \lambda\phi(\lambda^{(p-1)/2}r, \lambda^{p-1}t)$ satisfies

$$\phi_t^\lambda = \Delta\phi^\lambda + (\phi^\lambda)^p, \quad r < \lambda^{-(p-1)/2}r_1,\ 0 < t < \lambda^{-(p-1)}T_1, \tag{9.63}$$
$$\phi^\lambda(\lambda^{-(p-1)/2}r_1, t) = 0, \quad 0 < t < \lambda^{-(p-1)}T_1, \tag{9.64}$$
$$\phi^\lambda(\lambda^{-(p-1)/2}r_1, 0) = U(r,\lambda), \quad 0 < r < \lambda^{-(p-1)/2}r_1, \tag{9.65}$$

and blows up at $t = \lambda^{-(p-1)}T_1$. For $0 < \delta \ll 1$, we fix λ (it is clear that λ depends only on δ) such that

$$\lambda^{-(p-1)}T_1 = \delta, \quad \lambda^{-(p-1)/2}r_1 < R.$$

In view of Lemma 9.5 (c.f. (9.37)), $g_k(r) > U(|x-x^*|,\lambda)$ for $k \gg \lambda$ and $x^* \in \mathbb{R}^n$, $|x^*| \leqslant \varepsilon_0,\ \varepsilon_0 \ll 1$. Thus by using the maximum principle, we derive

$$\lim_{k\to\infty} v_k(r,t) \geqslant \phi^\lambda(|x-x^*|, t-T) \quad \text{for } T < t < T+\delta, \tag{9.66}$$

so that (notice that v_k is monotone nonincreasing in r)

$$\begin{aligned}\lim_{k\to\infty} \inf_{|x^*|\leqslant\varepsilon_0} v_k(r, T+\delta) &= \lim_{k\to\infty} v_k(\varepsilon_0, T+\delta) \\ &\geqslant \lim_{t\nearrow T+\delta} \phi^\lambda(0, t-T) = +\infty.\end{aligned} \tag{9.67}$$

Using the inequality,

$$v_k(|x|,t) \geqslant \int_{B_R(0)} G(x,y,t,T+\delta)v_k(|y|,\delta)dy, \qquad t > T+\delta. \tag{9.68}$$

We immediately obtain (as in Lemma 9.6)

$$\lim_{k\to\infty} v_k(r,t) = +\infty \quad \text{for } r < R-\delta, t > T+\delta. \qquad \square$$

Remark 9.5. In this argument we assume that 0 is a blow-up point. The condition (9.36) is assumed only for technical simplicity; this argument is valid without this assumption, with a more detailed analysis near the origin.

Remark 9.6. If the blow-up point is not at the origin, but on a ring $|x| = a > 0$, the argument needs to be modified using “travelling wave solutions” (c.f. [56]).

Remark 9.7. The stationary solutions behave differently when $p = (n+2)/(n-2)$. With a refined analysis, $p = (n+2)/(n-2)$ actually belongs to the complete blow-up case [56].

Remark 9.8. From the proof it is clear that the solution blows up in a neighborhood of the blow-up point $x = 0$ "immediately after" $t = T$. This should be the case if u is monotone in t, which is indeed true (c.f. Martel [100]):

Theorem 9.9. *Consider the problem*

$$\begin{aligned} &u_t - \Delta u = g(u) \geqslant 0, \quad x \in \Omega, \ \ t > 0, \\ &u = 0, \quad x \in \partial\Omega, \ \ t > 0, \\ &u(x,0) = u_0(x) \geqslant 0, \end{aligned}$$

where Ω is a bounded smooth domain. Suppose that u blows up in finite time. If $u_0 \in H_0^1(\overline{\Omega})$, and $\Delta u_0 + g(u_0) \geqslant 0$ in Ω, then the blow-up is complete.

Remark 9.9. There are examples (c.f. [56]) that the blow-up is incomplete for $\dfrac{n+2}{n-2} < p < 1 + \dfrac{6}{n-10}$ if $n \geqslant 11$, and $\dfrac{n+2}{n-2} < p < \infty$ if $n \leqslant 10$. The solution in the example blows up at $t = T$ and is bounded for each $t \neq T$.

We next outline how examples of incomplete blow-up solutions are constructed. We look for a solution in $\mathbb{R}^n$ of the form

$$\begin{aligned} u(x,t) &= (T-t)^{-1/(p-1)}\phi\Big(\frac{|x|}{\sqrt{T-t}}\Big) \quad \text{for } t < T, \\ u(x,t) &= (t-T)^{-1/(p-1)}\psi\Big(\frac{|x|}{\sqrt{t-T}}\Big) \quad \text{for } t > T. \end{aligned} \tag{9.69}$$

If we can find $\phi(y)$, $\psi(y)$ such that

$$\phi'' + \frac{n-1}{y}\phi' - \frac{1}{2}y\,\phi' - \frac{1}{p-1}\phi + \phi^p = 0, \quad \phi'(0) = 0, \tag{9.70}$$

$$\psi'' + \frac{n-1}{y}\psi' + \frac{1}{2}y\,\psi' + \frac{1}{p-1}\psi + \psi^p = 0, \quad \psi'(0) = 0, \tag{9.71}$$

$$\lim_{y\to\infty} \phi(y)y^{2/(p-1)} = \lim_{y\to\infty} \psi(y)y^{2/(p-1)} := c^* > 0, \tag{9.72}$$

then $u(x,t)$ is a blow-up solution that can be continued for all $t > T$. Under this construction, we have

$$\lim_{t\nearrow T} u(x,t) = \lim_{t\searrow T} u(x,t) = c^*|x|^{-2/(p-1)}. \tag{9.73}$$

Such a solution is called a *peaking solution*. The existence of a peaking solution is now reduced to the existence of ϕ and ψ satisfying (9.70)–(9.72).

Note that (9.70) and (9.71) are two separate problems, which we shall discuss in the next section.

9.4 Solutions in Similarity Variables

The analysis of the ordinary differential equations (9.70) and (9.71) can be very technical. Since they are closely related to the blow-up problems, it is not surprising that some blow-up estimates can be used in studying the properties of the solutions to (9.70) and (9.71).

In this section we shall only study (9.71).

Lemma 9.10. *Consider the solution $\psi(y,\lambda)$ of (9.71) with $\psi(0,\lambda)=\lambda>0$. There exists a $y_\lambda>0$ such that*

(a) The solution exists for $0<y<y_\lambda$, and

$$\psi(y,\lambda)>0 \quad \text{for } 0\leqslant y<y_\lambda, \qquad (\psi)_y(y,\lambda)<0 \quad \text{for } 0<y<y_\lambda;$$

(b) Either y_λ is infinite or y_λ is finite, in which case $\psi(y_\lambda,\lambda)=0$.

Proof. Integrating (9.71) we obtain

$$y^{n-1}e^{y^2/4}\psi'(y,\lambda)=-\int_0^y \xi^{n-1}e^{\xi^2/4}\Big(\frac{1}{p-1}\psi(\xi,\lambda)+\psi^p(\xi,\lambda)\Big)d\xi.$$

Here $'$ indicates the y derivative. Thus the solution exists for $0\leqslant y\ll 1$ and $\psi'<0$ as long as ψ exists and remains positive. This concludes the proof of the lemma. □

Lemma 9.11. *If $p\geqslant (n+2)/(n-2)$, then for any $\lambda>0$, $\psi(y,\lambda)$ exists for all $0\leqslant y<\infty$, i.e., $y_\lambda=\infty$.*

Proof. Suppose $y_\lambda<\infty$. We rewrite (9.71) as

$$\frac{d}{dy}\Big(y^{n-1}e^{y^2/4}\psi'(y,\lambda)\Big)+y^{n-1}e^{y^2/4}\Big(\frac{1}{p-1}\psi(y,\lambda)+\psi^p(y,\lambda)\Big)=0. \quad (9.74)$$

Multiplying (9.74) with ψ and integrating over $\{0<y<y_\lambda\}$, we obtain

$$\begin{aligned}&\int_0^{y_\lambda} e^{y^2/4}|\psi'(y,\lambda)|^2y^{n-1}dy\\ &\quad=\int_0^{y_\lambda} e^{y^2/4}\Big(\frac{1}{p-1}\psi^2(y,\lambda)+\psi^{p+1}(y,\lambda)\Big)y^{n-1}dy.\end{aligned} \quad (9.75)$$

Multiplying (9.74) with $y^2\psi$ and integrating over $\{0<y<y_\lambda\}$, we get

$$\begin{aligned}
&\int_0^{y_\lambda} e^{y^2/4} y^2 |\psi'(y,\lambda)|^2 y^{n-1} dy \\
&= \int_0^{y_\lambda} e^{y^2/4} y^2 \psi^{p+1}(y,\lambda) y^{n-1} dy \\
&\quad + \int_0^{y_\lambda} e^{y^2/4} \Big\{ n\, \psi^2(y,\lambda) + \frac{p+1}{2(p-1)} y^2 \psi^2(y,\lambda) \Big\} y^{n-1} dy.
\end{aligned} \tag{9.76}$$

Finally, multiplying (9.74) with $y\,\psi'$ and integrating over $\{0 < y < y_\lambda\}$, we derive

$$\begin{aligned}
&\int_0^{y_\lambda} e^{y^2/4} \Big(\frac{y^2}{4} + \frac{n-2}{2} \Big) |\psi'(y,\lambda)|^2 y^{n-1} dy + \frac{1}{2} y_\lambda^n e^{y_\lambda^2/4} |\psi'(y_\lambda,\lambda)|^2 \\
&= \int_{\mathbb{R}^n} \Big(\frac{y^2}{2} + n \Big) e^{y^2/4} \cdot \Big(\frac{1}{p+1} \psi^{p+1}(y,\lambda) + \frac{1}{2(p-1)} \cdot \psi^2(y,\lambda) \Big) dy.
\end{aligned} \tag{9.77}$$

The combination $2n \times (9.75) + (9.76) - 2(p+1) \times (9.77)$ gives

$$\begin{aligned}
&\int_0^{y_\lambda} e^{y^2/4} \Big((n+2) - (n-2)p - \frac{p-1}{2} y^2 \Big) |\psi'(y,\lambda)|^2 y^{n-1} dy \\
&\qquad -(p+1)\, y_\lambda^n e^{y_\lambda^2/4} |\psi'(y_\lambda,\lambda)|^2 = 0,
\end{aligned}$$

which is a contradiction since $p \geqslant (n+2)/(n-2)$. ◻

Lemma 9.12. *Suppose that* $p \geqslant (n+2)/(n-2)$ *and* ψ *is the positive solution to* (*9.71*) *for all* $y \geqslant 0$. *Then*

$$\liminf_{y\to\infty} \Big(-\frac{\psi'(y,\lambda)}{\psi(y,\lambda)} \Big) \neq +\infty.$$

Proof. If the conclusion is not true, then

$$-\frac{\psi'(y,\lambda)}{\psi(y,\lambda)} > 1 \quad \text{for } y \gg 1,$$

and therefore

$$0 < \psi(y,\lambda) < Ce^{-y} \quad \text{for } y \gg 1.$$

In particular, ψ and their derivatives are integrable near ∞. The function

$$u(x,t) = (t+1)^{-1/(p-1)} \psi\Big(\frac{|x|}{\sqrt{t+1}} \Big) \quad \text{for } t > 0.$$

is a global solution to the system

$$\begin{aligned}
&u_t - \Delta u = u^p \quad \text{in } \mathbb{R}^n \times (0,\infty), \\
&u(x,0) = \psi(|x|).
\end{aligned}$$

By the estimates at ∞ for ψ and its derivatives, u and its spatial derivatives are all integrable near ∞.

Since our solution $u(x,t)$ is global, we must have, by Theorem 5.3,

$$\begin{aligned} E(t) &:= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla_x u(x,t)|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^n} u^{p+1}(x,t)dx \\ &\geqslant 0 \quad \text{for all } 0 < t < \infty. \end{aligned} \tag{9.78}$$

We can compute as in Chap. 8 that

$$E'(t) = -\int_{\mathbb{R}^n} |u_t(x,t)|^2 dx. \tag{9.79}$$

A direct computation shows

$$\begin{aligned} E(t) &= (t+1)^{\{(n-2)p-(n+2)\}/\{2(p-1)\}} \\ &\quad \times \Big(\frac{1}{2}\int_{\mathbb{R}^n} |\psi'(|\xi|)|^2 d\xi - \frac{1}{p+1}\int_{\mathbb{R}^n} \psi^{p+1}(|\xi|)d\xi\Big). \end{aligned}$$

Recalling (9.78) and $p > (n+2)/(n-2)$ we find that

$$E'(t) \geqslant 0 \quad \text{for } 0 < t < \infty. \tag{9.80}$$

This is a contradiction to (9.79) since $u_t \not\equiv 0$. $\square$

Lemma 9.13. *Suppose $p \geqslant (n+2)/(n-2)$ and the solution ψ to (9.71) is positive for all $y \geqslant 0$. Then the limit*

$$\lim_{y\to\infty} \psi(y,\lambda)y^{2/(p-1)} := c_\lambda$$

exists and is positive.

Proof. The function $z(y) = -\dfrac{\psi'(y,\lambda)}{\psi(y,\lambda)}$ satisfies

$$z' = z^2 - \Big(\frac{n-1}{y} + \frac{1}{2}y\Big)z + \frac{1}{p-1} + \psi^{p-1}(y,\lambda) \tag{9.81}$$

Note that for any $\alpha > 0$, there exists $K \gg 1$ such that $z'(y) < 0$ on the set $\{y > K;\ z(y) = \alpha\}$ (note that ψ is monotone decreasing in y). If there exists $y_0 > K$ such that $z(y_0) < \alpha < +\infty$, then we must have $z(y) < \alpha$ for all $y > y_0$. Thus

$$\limsup_{y\to\infty} z(y) \leqslant \liminf_{y\to\infty} z(y).$$

By Lemma 9.12 we have $\lim_{y\to\infty} z(y)$ is finite. Since, for any $\alpha > 0$, $z'(y)$ is strictly negative on the set $\{y \gg 1;\ z(y) = \alpha\}$, we must have

$$\lim_{y\to\infty} z(y) = 0.$$

From (9.81),

$$yz(y) = \frac{\displaystyle\int_0^y \xi^{n-1} e^{\xi^2/4}\Big(\frac{1}{p-1} + \psi^{p-1}(\xi,\lambda) + z^2(\xi)\Big)d\xi}{y^{n-2}e^{y^2/4}}. \tag{9.82}$$

We now the L'Hospital's rule to conclude

$$\lim_{y\to+\infty} \frac{-y\psi'(y,\lambda)}{\psi(y,\lambda)} = \lim_{y\to+\infty} yz(y) = \frac{2}{p-1};$$

this implies that, for any $0 < \delta_1 < 2/(p-1)$, $y^{\delta_1}\psi(y,\lambda)$ is monotone decreasing for sufficiently large y, and the limit $\lim_{y\to\infty} y^{\delta_1}\psi(y,\lambda)$ exists; thus for $0 < \delta < \delta_1$,

$$\lim_{y\to\infty} y^{\delta}\psi(y,\lambda) = 0. \tag{9.83}$$

Taking $0 < \varepsilon < \delta(p-1)$, using (9.82) and L'Hospital's rule, we derive

$$\begin{aligned}
&\lim_{y\to+\infty} y^{\varepsilon}\Big(\frac{2}{p-1} + \frac{y\psi'(y,\lambda)}{\psi(y,\lambda)}\Big)\\
&= \lim_{y\to+\infty} \frac{\frac{2}{p-1}y^{n-2}e^{y^2/4} - \displaystyle\int_0^y \xi^{n-1}e^{\xi^2/4}\Big(\frac{1}{p-1} + \psi^{p-1}(\xi,\lambda) + z^2(\xi)\Big)d\xi}{y^{n-2-\varepsilon}e^{y^2/4}}\\
&= 0.
\end{aligned}$$

This estimate implies that $w = \log\Big\{y^{2/(p-1)}\psi(y,\lambda)\Big\}$ satisfies $|w'(y)| \leqslant Cy^{-1-\varepsilon}$ for $y \gg 1$, so that $w(y)$ converges to a finite number as $y \to \infty$. Thus

$$\lim_{y\to+\infty} y^{2/(p-1)}\psi(y,\lambda) = c_\lambda > 0. \qquad \square$$

Lemma 9.14. *Suppose $p \geqslant (n+2)/(n-2)$ and the solution ψ to (9.71) is positive for all $y \geqslant 0$. Then c_λ defined in Lemma 9.13 depends continuously on λ, and $\lim_{\lambda\searrow 0} c_\lambda = 0$.*

Proof. From the ODE it is clear that for any finite $a \geqslant 1$, $\psi(a,\lambda), \psi_y(a,\lambda)$ depends continuously on λ. Using the equation one can show that $y^{\varepsilon}\psi(a,\lambda)$ $(0 < \varepsilon \ll 1)$ is bounded on $[1,\infty)$, uniformly in a neighborhood of any $\lambda = \lambda_0$.

The function $w(y,\lambda) = y^{2/(p-1)}\psi(y,\lambda)$ satisfies

$$w'' + \Big(a_1 y^{-1} + \frac{1}{2}y\Big)w' = a_2 y^{-2} w - y^{-2} w^p := J[w](y), \tag{9.84}$$

where

$$a_1 = n - 1 - \frac{4}{p-1} > 0, \quad a_2 = \frac{2}{p-2}\Big((n-2) - \frac{2}{p-1}\Big) > 0.$$

Thus

$$y^{a_1} e^{y^2/4} w_y(y,\lambda) = y_1^{a_1} e^{y_1^2/4} w_y(y_1,\lambda) + \int_{y_1}^{y} \xi^{a_1} e^{\xi^2/4} J[w](\xi)d\xi, \tag{9.85}$$

and, for $y > y_1$,

$$\begin{aligned} w(y,\lambda) - w(y_1,\lambda) &= y_1^{a_1} e^{y_1^2/4} w_y(y_1,\lambda) \int_{y_1}^{y} \frac{1}{s^{a_1} e^{s^2/4}} ds \\ &\quad + \int_{y_1}^{y} \frac{1}{s^{a_1} e^{s^2/4}} \int_{y_1}^{s} \xi^{a_1} e^{\xi^2/4} J[w](\xi) d\xi ds \\ &= y_1^{a_1} e^{y_1^2/4} w_y(y_1,\lambda) \int_{y_1}^{y} \frac{1}{s^{a_1} e^{s^2/4}} ds \\ &\quad + \int_{y_1}^{y} J[w](\xi) \Big(\int_{\xi}^{y} \Big(\frac{\xi}{s}\Big)^{a_1} e^{(\xi^2 - s^2)/4} ds \Big) d\xi. \end{aligned} \tag{9.86}$$

Notice that

$$\int_{\xi}^{y} \Big(\frac{\xi}{s}\Big)^{a_1} e^{(\xi^2-s^2)/4} ds \leqslant \int_{\xi}^{y} \frac{s}{\xi} e^{(\xi^2-s^2)/4} ds \leqslant \frac{2}{\xi}. \tag{9.87}$$

It is clear that

$$\lim_{\lambda \to \lambda_0} J[w(\cdot,\lambda)](y) \to J[w(\cdot,\lambda_0)](y) \text{ for each } 0 < y < \infty.$$

Letting $y \to \infty$ in (9.86) and using Lebesgue's dominated convergence theorem, we conclude c_λ is continuous in λ. Since $\psi(y,\lambda) \equiv 0$ when $\lambda = 0$, we derive that c_λ is small for positive small λ. □

Lemmas 9.10–9.14 can be summarized as follows:

Theorem 9.15. *Let $p \geqslant (n+2)/(n-2)$. Then the nontrivial solution ψ to (9.71) is positive for all $y \geqslant 0$. Furthermore the limit*

$$\lim_{y \to \infty} \psi(y,\lambda) y^{2/(p-1)} := c_\lambda$$

exists and is positive, c_λ depends continuously on λ, and c_λ is small for small λ.

Remark 9.10. For the solution $\phi(y,\lambda)$ to (9.70) with $\phi(0,\lambda)=\lambda$:

$$\phi'' + \frac{n-1}{y}\phi' - \frac{1}{2}y\,\phi' = \frac{1}{p-1}\phi - \phi^p, \quad \phi'(0)=0.$$

The singular solution defined by (9.51) satisfies

$$S'' + \frac{n-1}{y}S' + S^p = \frac{1}{p-1}S + \frac{1}{2}y\,S' = 0.$$

There is also a positive constant solution

$$\kappa = \left(\frac{1}{p-1}\right)^{1/(p-1)}.$$

The analysis uses the intersection between $\phi(y,\lambda)$ and $S(y)$. The study relies on this intersection property established in Joseph–Lundgren [79] for $\frac{n+2}{n-2} < p < 1+\frac{4}{n-4-2\sqrt{n-1}}$. In this range of p, it is established [56] that there are solutions which intersect $S(y)$ exactly $2m$ ($m=1,2,3,\cdots$) times such that $y^{2/(p-1)}\phi(y,\lambda)$ has a limit as $y\to\infty$. Peaking solutions are then constructed by solving (9.72).

9.5 Exercises

9.1. Theorem 9.1 deals with the Dirichlet Boundary condition. State the result for the Neumann boundary data and prove it.

9.2. For (9.39) and (9.40), show that the number of sign changes of $J(t)$ is nonincreasing in time.

9.3. Prove Theorem 9.8 for the case $p=(n+2)/(n-2)$ by the following procedure:

(a) For any fixed $\delta>0$ show that

$$\inf\{u(r,t);\ \ r\leqslant R-\delta,\ \delta<t<T\}\geqslant c>0.$$

(b) Show that the solution of (9.29) and (9.30) is defined for all $0\leqslant r<\infty$, and

$$U(r) = O(r^{-(n-2)}) \quad \text{as } r\to\infty.$$

(c) Show that, for any $\delta>0$,

$$\sup_{\delta<r<\infty} U(r,\lambda)\to 0 \quad \text{as } \lambda\to\infty.$$

(d) Show that for $\lambda \gg 1, U(r,\lambda)$ will intersect $u(x,t)$ at most twice, and that one of the intersections lies in $\{R-\delta < r < R\}$.

(e*) For the system

$$\begin{aligned}
&\phi_t + \Delta\phi = \phi^p, \quad r < r_1, \ \ t > 0,\\
&\phi(r_1, t) = c > U(r_1, \lambda), \quad t > 0,\\
&\phi(r, 0) = U(r, \lambda),
\end{aligned}$$

($\lambda \gg 1$) establish the finite time blow-up result and estimate the blow-up time in terms of λ. The fact $p = (n+2)/(n-2)$ is used here.

(f) Complete the proof.

References

1. R.A. Adams, Sobolev Spaces, Academic, New York, 1975
2. M.M. Ad'jutov and L.A. Lepin, *Absence of blowing up similarity structures in a medium with source for constant thermal conductivity*, Differ. Uravn. 20 (1984), 1279–1281
3. J.R. Anderson and K. Deng, *Global existence for nonlinear diffusion equations*, J. Math. Anal. Appl. 196 (1995), 479–501
4. S. Angenent, *The zero set of a solution of a parabolic equation.* J. Reine Angew. Math. 390 (1988), 79–96
5. H. Amann and M. Fila, *A Fujita-type theorem for the Laplace equation with a dynamical boundary condition*, Acta Math. Uni. Comenian. 66 (1997), 321–328
6. D. Aronson and H.F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. Math. 30 (1978), 33–76
7. J. Ball, *Remarks on blow-up and nonexistence theorems for nonlinear evolution equations*, Quart. J. Math. Oxford. 28 (1977), 473–486
8. P. Baras and L. Cohen, *Complete blow-up after $T_{\max}$ for the solution of a semilinear heat equation*, J. Funct. Anal. 71 (1987), no. 1, 142–174
9. G.I. Barenblatt, Similarity, Self-similarity, and intermediate asymptotics, Consultants Bureau, New York, 1979
10. C. Bandle and H. Brunner, *Blow-up in diffusion equations, a survey*, J. Comp. Appl. Math. 97 (1998), 3–22
11. C. Bandle and H.A. Levine, *On the existence and nonexistence of global solutions of reaction-diffusion equations in sectorial domains*, Trans. Am. Math. Soc. 316 (1989), 595–622
12. C. Bandle and H.A. Levine, *Fujita type phenomena for reaction-diffusion equations with convection like terms*, Differ. Integr. Equat. 7 (1994), 1169–1193
13. J. Bebernes and V.A. Galaktionov, *On classification of blow-up patterns for a quasilinear heat equation*, Differ. Integr. Equat. 9 (1996), no. 4, 655–670
14. C.J. Budd and V.A. Galaktionov, *Critical diffusion exponents for self-similar blow-up solutions of a quasilinear parabolic equation with an exponential source*, Proc. Roy. Soc. Edinburgh Sect. A 126 (1996), no. 2, 413–441
15. J. Bebernes and D. Eberly, Mathematical Problems from Combustion Theory, Springer, New York, 1989
16. L.A. Caffarelli, B. Gidas and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. 42 (1989), 271–297
17. L.A. Caffarelli and A. Friedman, *Blow-up of solutions of nonlinear heat equations*, J. Math. Anal. Appl. 129 (1988), 409–419
18. W. Chen and C. Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. 63 (1991), 615–622
19. X.-Y. Chen and H. Matano, *Convergence, asymptotic periodicity and finite-point blow up in one dimensional semilinear heat equations*, J. Differ. Equat. 78 (1989), 160–190
20. Y.-Z. Chen, Second order parabolic equations (in Chinese), Beijing University Mathematics series, Beijing University Press, Beijing, 2003

B. Hu, *Blow-up Theories for Semilinear Parabolic Equations,* Lecture Notes in Mathematics 2018, DOI 10.1007/978-3-642-18460-4,

21. Y.-Z. Chen and L.-C. Wu, Second order elliptic equations and elliptic systems, Vol. 174 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1998. Translated from the 1991 Chinese original by Bei Hu
22. M. Chipot, M. Fila and P. Quittner, *Stationary solutions, blow-up and convergence to stationary solutions for semilinear parabolic equations with nonlinear boundary conditions*, Acta Math. Univ. Comenian. LX(1991), 35–103
23. M. Chipot, I. Shafrir and M. Fila, *On the solutions to some elliptic equations with nonlinear neumann boundary conditions*, Adv. Differ. Equat. 1 (1996), 91–110
24. M. Chlebik and M. Fila, *From critical exponents to blow-up rates for parabolic problems,* Rend. Mat. Appl. 19 (1999), 449–470
25. K. Deng, *Nonexistence of global solutions of a nonlinear hyperbolic system*, Trans. Am. Math. Soc. 349 (1997), 1685–1696
26. K. Deng, *Blow-up of solutions of some nonlinear hyperbolic systems*, Rocky Mountain J. Math. 29 (1999), 807–820
27. K. Deng, M. Fila and H.A. Levine, *On critical exponent for a system of heat equations coupled in the boundary conditions*, Acta Math. Univ. Comenian. 63 (1994), 169–192
28. K. Deng and H.A. Levine, *The role of critical exponents in blow-up theorems: the sequel*, J. Math. Anal. Appl. 243 (2000), 85–126
29. G. Dong, Nonlinear partial differential equations of second order, Vol. 95 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1991. Translated from the 1988 Chinese original by Kai Seng Chou
30. L.C. Evans, Partial differential equations, Graduate Studies in Mathematics, Vol 19, American Mathematical Society, Providence, RI, 1991
31. M. Escobedo and M.A. Herrero, *Boundedness and blow up for a semilinear reaction-diffusion system*, J. Differ. Equat. 89 (1991), 176–202
32. M. Escobedo and H.A. Levine, *Critical blow-up and global existence numbers for a weakly coupled system of reaction-diffusion equations*, Arch. Rational Mech. Anal. 129 (1995), 47–100
33. M. Fila, *Boundedness of global solutions for the heat equation with nonlinear boundary conditions*, Comment. Math. Univ. Carolin. 30 (1989) 479–484
34. M. Fila and J. Filo, *Blow-up on the boundary: a survey*, Proceedings of the Banach Center, Vol. 33, Polish Academy of Science, Institute of Mathematics, Warsaw, 1996, pp. 67–78
35. M. Fila and J. Filo, *A blow-up result for nonlinear diffusion equations*, Math. Slovaca 39 (1989), no. 3, 331–346
36. M. Fila and H.A. Levine, *On critical exponents for a semilinear parabolic system coupled in an equation and a boundary condition*, J. Math. Anal. Appl. 204 (1996), 494–521
37. M. Fila, H.A. Levine and Y. Uda, *A Fujita-type global existence-global non-existence theorem for a system of reaction-diffusion equations with differing diffusivities*, Math. Meth. Appl. Sci. 17 (1994), 807–835
38. M. Fila and P. Quittner, *The blow-up rate for the heat equation with a nonlinear boundary condition*, Math. Meth. Appl. Sci. 14 (1991), 197–205
39. S. Filippas, A.M. Herrero and J.J.L. Velazquez, *Fast blow-up mechanisms for sign-changing solutions of a semilinear parabolic equation with critical nonlinearity*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 456 (2000), 2957–2982
40. J. Filo, *Uniform bounds for solutions of a degenerate diffusion equation with nonlinear boundary conditions*, Comment. Math. Univ. Carolin. 30 (1989), 485–495
41. J. Filo, *Diffusivity versus Absorption through the Boundary*, J. Differ. Equat. 99 (1992), 281–305
42. A. Friedman and B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J. 34 (1985), 425–477
43. A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, NJ, 1964
44. S.C. Fu and J.S. Guo, *Blow-up for semilinear reaction-diffusion system coupled in both euqations and boundary conditions*, J. Math. Appl. Anal. 276 (2002), 458–475

45. S.C. Fu, J.S. Guo and J.C. Tsai, *Blow-up behavior for a semilinear heat equation with a nonlinear boundary condition*, Tohoku Math. J. (2) 55 (2003), no. 4, 565–581
46. H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. A. Math. 16 (1966), 105–113
47. V.A. Galaktionov, *On conditions for there to be no global solutions of a class of quasilinear parabolic equations*, USSR Comp. Math. Math. Phys. 22 (1982), 73–90
48. V.A. Galaktionov, *On a blow-up set for the quasilinear heat equation* $u_t = (u^\sigma u_x)_x + u^{\sigma+1}$, J. Differ. Equat. 101 (1993), no. 1, 66–79
49. V.A. Galaktionov, *Blow-up for quasi-linear heat equations with critical Fujita's exponents*, Proc. Roy. Soc. Edinburgh Ser. A 124 (1994), 517–525
50. V.A. Galaktionov, *On asymptotic self-similar behavior for quasilinear heat equation: single point blow-up*, SIAM J. Math. Anal. 26 (1995), 675–693
51. V.A. Galaktionov, Geometric Sturmian theory of nonlinear parabolic equations and applications, Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series, 3. Chapman & Hall/CRC, Boca Raton, FL, 2004
52. V.A. Galaktionov and H.A. Levine, *On critical Fujita exponents for heat equations with a nonlinear flux boundary condition on the boundary*, Israel J. Math, 94 (1996), 125–146
53. V.A. Galaktionov and H.A. Levine, *A general approach to critical Fujita exponents and systems*, Nonlinear Anal. 34 (1998), 1005–1027
54. V.A. Galaktionov and S.I. Pohozaev, *Blow-up and critical exponents for nonlinear hyperbolic equations*, Nonlinear Anal. 53 (2003), 453–466
55. V.A. Galaktionov and J.L. Vàzquez, *Necessary and sufficient conditions of complete blow up and extinction for one-dimensional quasilinear heat equations*, Arch. Rational Mech. Anal. 129 (1995), 225–244
56. V.A. Galaktionov and J.L. Vàzquez, *Continuation of blow-up solutions of nonlinear heat equations in several space dimensions*, Comm. Pure Appl. Math. 50 (1997), 1–67
57. V.A. Galaktionov and J.L. Vàzquez, *Incomplete blow-up and singular interfaces for quasilinear heat equations*, Comm. Partial Differ. Equat. 22 (1997), no. 9–10, 1405–1452
58. V.A. Galaktionov and J.L. Vàzquez, *A stability technique for evolution partial differential equations*, Birkhäuser, Berlin, 2004
59. M. Giaquinta, Introduction to Regularity Theorey for Nonlinear Elliptic Systems, Birkhäuser, Berlin, 1993
60. B. Gidas, W.N. Ni and L. Nirenberg, *Symmetry and related problems via the maximum principle*, Comm. Math. Phys. 68 (1979), 209–243
61. B. Gidas, W.M. Ni and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^n$ (collected in the book Mathematical Analysis and Applications, Vol. 7a, the Advances in Mathematics, Supplementary Studies), Academic, New York, 1981
62. B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure. Appl. Math. 34 (1981), 525–598
63. B. Gidas and J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Partial Differ. Equat. 8 (1981), 883–901
64. Y. Giga and R.V. Kohn, *Asymptotic self-similar blow-up of semilinear heat equations*, Comm. Pure. Appl. Math. 38 (1985), 297–319
65. Y. Giga and R.V. Kohn, *Characterizing blow-up using similarity variables*, Indiana Univ. Math. J. 36 (1987), 425–447
66. Y. Giga and R.V. Kohn, *Nondegeneracy of blow-up for semilinear heat equations*, Comm. Pure Appl. Math. 42 (1989), 845–884
67. D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Third edition, Springer, New York, 1998
68. J.S. Guo and B. Hu, *Blow-up rate for heat equation in Lipschitz domains with nonlinear heat source terms on the boundary*, J. Math. Anal. Appl. 269 (2002), 28–49
69. T. Hamada, *Nonexistence of global solutions of parabolic equation in conical domains*, Tsukuba J. Math. 19 (1995), 15–25

70. K. Hayakawa, *On the nonexistence of global solutions of some semilinear parabolic equations*, Proc. Jpn. Acad. 49 (1973), 503–525
71. M.A. Herrero and J.J.L. Velazquez, *Explosion des solutions d'équations paraboliques semilinéaires*, C. R. Acad. Sci. Paris 319 (1994), 141–145
72. Bei Hu, *Nonexistence of a positive solution of the Laplace equation with a nonlinear boundary condition*, Differ. Integr. Equat. 7 (1994), 301–313
73. Bei Hu, *Remarks on the blow-up estimate for solution of the heat equation with a nonlinear boundary condition*, Differ. Integr. Equat. 9 (1996), 891–901
74. Bei Hu, *Nondegeneracy and single-point-blow-up for solution of the heat equation with a nonlinear boundary condition*, J. Math. Sci. Univ. Tokyo 1 (1994), 251–278
75. Bei Hu and H.-M. Yin, *The profile near blow-up time for solution of the heat equation with a nonlinear boundary condition*, Trans. Am. Math. Soc. 346 (1994), 117–135
76. Bei Hu and H.-M. Yin, *On critical exponents for the heat equation with a nonlinear boundary condition*, Ann. Inst. H. Poincaré 13 (1996), 707–732
77. Bei Hu and H.-M. Yin, *On critical exponents for the heat equation with a mixed nonlinear Dirichlet-Neumann boundary condition*, J. Math. Anal. Appl. 209 (1997), 683–711
78. Bei Hu and H.-M. Yin, *Critical exponents for a system of heat equations coupled in a non-linear boundary condition*, Math. Meth. Appl. Sci. 19 (1996), 1099–1120
79. D.D. Joseph and T.S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal. 49, (1972/73), 241–269
80. S. Kaplan, *On the growth of solutions of quasilinear parabolic equations*, Comm. Pure Appl. Math. 16 (1963), 305–333
81. V.K. Kalantarov and O.A. Ladyzhenskaya, *Formation of collapses in quasilinear equations of parabolic and hyperbolic types*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 69 (1977), 77–102
82. K. Kobayashi, T. Siaro and H. Tanaka, *On the blow-up problem for semilinear heat equations*, J. Math. Soc. Jpn. 29 (1977), 407–424
83. O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Ural'ceva, Linear and Quasi-linear Equations of Parabolic Type, AMS Trans. 23, Providence, RI, 1968
84. O.A. Ladyzenskaja and N.N. Ural'ceva, Linear and Quasi-linear Elliptic Equations, Mathematics in Sciences and Engineering, Academic, New York, 1968
85. H.A. Levine, *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form* $Pu_t = -Au + F(u)$, Arch. Rational Mech. Anal. 51 (1973), 371–386
86. H.A. Levine, *The role of critical exponents in blow-up theorems*, SIAM Rev. 32 (1990), 262–288
87. H.A. Levine, *A. Fujita type global existence-global nonexistence theorem for a weakly coupled system of reaction-diffusion equations*, Z. Angew. Math. Phys. 42 (1991), 408–430
88. H.A. Levine and P. Meier, *A blow-up result for the critical exponent in cones*, Israel J. Math. 67 (1989), 1–7
89. H.A. Levine and P. Meier, *The value of the critical exponent for reaction-diffusion equations in cones*, Arch. Rational Mech. Anal. 109 (1990) 73–80
90. H.A. Levine and L.E. Payne, *Nonexistence theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time*, J. Differ. Equat. 16 (1974), 319–334
91. H.A. Levine and L.E. Payne, *Some nonexistence theorems for initial-boundary value problems with nonlinear boundary constraints*, Proc. Am. Math. Soc. 46 (1974), 277–284
92. H.A. Levine and R.A. Smith, *A potential well theory for the heat equation with a nonlinear boundary condition*, Math. Meth. Appl. Sci. 9 (1987), 127–136
93. G.M. Lieberman, *Intermediate Schauder estimates for oblique derivative problems*, Arch. Rational Mech. Anal., 93 (1986), 129–134
94. G.M. Lieberman, Second Order Parabolic Differential Equations, World Scientific, Singapore, 1996

95. G. Lieberman, *Study of global solutions of parabolic equations via a priori estimates. Part I: Equations with principal elliptic part equal to the Laplacian*, Math. Meth. Appl. Sci. 16 (1993), 457–474
96. Y.Y. Li and M. Zhu, *Uniqueness theorems through the method of moving spheres*, Duke. Math. J. 80 (1995), 383–417
97. W. Liu, *The blow-up rate of solutions of semilinear heat equations*, J. Differ. Equat. 77 (1989), 104–122
98. J.L. Gómez, V. Márquez and N. Wolanski, *Blow up results and localization of blow up points for the heat equation and localization of blow up points for the heat equation with a nonlinear boundary condition*, J. Differ. Equat. 92 (1991), 384–401
99. G. Lu and B.D. Sleeman, *Sub and super solutions to systems of parabolic equations with applications to generalized Fujita type systems*, Math. Meth. Appl. Sci. 17 (1994), 1005–1016
100. Y. Martel, *Complete blow up and global behaviour of solutions of* $u_t - \Delta u = g(u)$, Ann. Inst. H. Poincaré, 15 (1998), 687–723
101. H. Matano, *Nonincrease of the lap number of a solution for a one dimensional semilinear parabolic equation*, J. Fac. Sci. Univ. Tokyo, Sect. 1A Math. 29 (1982), 401–441
102. H. Matano and F. Merle, *On non-existence of Type II blow-up for supercritical nonlinear heat equation*, Comm. Pure. Appl. Math. 57 (2004), 1494–1541
103. P. Meier, *Blow-up of solutions of semilinear parabolic differential equations*, Z. Angew. Math. Phys. 39 (1988), 135–149
104. P. Meier, *On the critical exponent for reaction-diffusion equations*, Arch. Rational Mech. Anal. 109 (1990), 63–72
105. F. Merle, *Solution of a nonlinear heat equation with arbitrary given blow-up points*, Comm. Pure Appl. Math. 45 (1992), 263–300
106. N. Mizoguchi, *Type II blow-up for semilinear heat equation*, Adv. Differ. Equat. 9 (2004), 1279–1316
107. N. Mizoguchi and E. Yanagida, *Critical exponents for the blow-up of solutions with sign changes in a semilinear parabolic equation*, Math. Ann. 307 (1997), 663–675
108. N. Mizoguchi and E. Yanagida, *Critical exponents for the blow-up of solutions with sign changes in a semilinear parabolic equation II*, J. Differ. Equat. 145 (1998), 295–331
109. K. Mochizuki and Q. Huang, *Existence and behavior of solutions for a weakly coupled system of reaction-diffusion equations*, Meth. Appl. Math. 5 (1998), 109–124
110. K. Mochizuki and K. Mukai, *Existence and nonexistence of global solutions to fast diffusions with source*, Math. Meth. Appl. Anal. 2 (1995), 92–102
111. K. Mochizuki and R. Suzuki, *Critical exponent and critical blow up for quasi-linear parabolic equations*, Israel J. Math. 98 (1997), 141–156
112. S. Ohta and A. Kaneko, *Critical exponent of blow-up for semilinear heat equation on a product domain*, J. Fac. Sci. Univ. Tokyo, Sec. 1A Math. 40 (1994), 635–650
113. W.F. Osgood, *Beweis der Existenz einer Lösung der Differentialgleichung* $dy/dx = f(x, y)$ *ohne Hinzunahme der Cauchy-Lipschitzschen Bedingung*, Monatshefte der Mathemratik unid Physik, 9 (1898), 331–345
114. L.E. Payne, *Improperly posed problems in partial differential equations*, Regional Conference Series in Applied Mathematics, No. 22. SIAM, Philadelphia, PA, 1975
115. L.E. Payne and D.H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations*, Israel J. Math. 22 (1975), 273–303
116. R.G. Pinsky, *Existence and nonexistence of global solutions for* $u_t = \Delta u + a(x)u^p$ *in* $\mathbb{R}^d$, J. Differ. Equat. 133 (1997), 152–177
117. R.G. Pinsky, *Finite time blow-up for the inhomogeneous equation* $u_t = \Delta u + a(x)u^p + \lambda\phi$ *in* $\mathbb{R}^d$, Proc. Am. Math. Soc. 127 (1999), 3313–3327
118. M. Protter and H.F. Weinberger, Maximum Principles in Differential Equations, Prentice-Hall, Englewood Cliffs, NJ, 1967
119. Y.W. Qi, *On the equation* $u_t = \Delta u^\alpha + u^\beta$, Proc. Roy. Soc. Edinburgh Ser. A 123 (1993), 373–390

120. Y.W. Qi, *The critical exponents of parabolic equations and blow-up in* $\mathbb{R}^n$, Proc. Roy. Soc. Edinburgh Ser. A 128 (1998), 123–136
121. Y.W. Qi and H.A. Levine, *The critical exponent of degenerate parabolic systems*, Z. Angew. Math. Phys. 44 (1993), 249–265
122. P. Quittner, *On global existence and stationary solutions for two classes of semilinear parabolic problems*, Comment. Math. Univ. Carolin. 34 (1993), 105–124
123. P. Quittner and P. Souplet, Superlinear Parabolic Problems, Birkhäuser, Berlin, 2007
124. J. Renclawowicz, *Global existence and blow up for a completely coupled Fujita type system of reaction-diffusion equations*, Appl. Math. 25 (1998), 313–326
125. A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov and A.P. Mikhailov, Blow-up in quasilinear parabolic equations, Translated from the 1987 Russian original by Michael Grinfeld and revised by the authors, Walter de Gruyter & Co, Berlin, 1995
126. P. Souplet, *Uniform blow-up profiles and boundary behavior for diffusion equations with nonlocal nonlinear source*, J. Differ. Equat. 153 (1999), 374–406
127. P. Souplet and W. Weissler, *Self-similar subsolutions and blow-up for nonlinear parabolic problems*, J. Math. Anal. Appl. 212 (1997), 60–74
128. C. Sturm, *Mémorie sur une classe d'équations á différences partielles*, J. Math. Pure. Appl. 1 (1836), 373–444
129. R. Suzuki, *Critical Blow up for quasilinear parabolic equations in exterior domains*, Tokyo J. Math. 19 (1996), 397–409
130. R. Suzuki, *Existence and nonexistence of global solutions to quasilinear parabolic equations with convection*, Hokkaido Math. J. 27 (1998), 147–196
131. M. Tsutsumi, *Existence and nonexistence of global solutions for nonlinear parabolic equations*, Publ. Res. Inst. Math. Sci. Kyoto Univ. 8(1972), 221–229
132. Y. Uda, *The critical exponent for a weakly coupled system of the generalized Fujita type reaction-diffusion equations*, Z. Angew Math. Phys. 46 (1995), 366–383
133. W. Walter, *On existence and nonexistence in the large of solutions of parabolic differential equations with a nonlinear boundary condition*, SIAM J. Math. Anal. 6 (1975), 85–90
134. F. Weissler, *Existence and nonexistence of global solutions of a semi-linear heat equation*, Israel J. Math. 38 (1981), 29–40
135. F. Weissler, *Single point blow-up for a semilinear initial value problem* J. Differ. Equat. 55 (1984), 204–224
136. M.X. Wang, *Blow-up estimates for semilinear parabolic systems coupled in an equation and a boundary condition*, Sci. China Ser A. 44 (2001), 1465–1468
137. M.X. Wang, *Blow-up properties of solutions to parabolic systems coupled in an equation and a boundary condition*, J. Lond. Math. Soc. 67 (2003), 180–194
138. M.X. Wang, *Blow-up rate estimates for semilinear parabolic systems*, J. Differ. Equat. 170 (2001), 317–324
139. M.X. Wang and Y. Wu, *Global existence and blow-up problems for quasilinear parabolic equations with nonlinear boundary conditions*, SIAM J. Math. Anal. 24 (1993), 1515–1521
140. N. Wolanski, *Global behavior of positive solutions to nonlinear diffusion problems with nonlinear absorption through the boundary*, SIAM J. Math. Anal. 24 (1993), 317–326
141. H.-M. Yin, Blow-up versus global solvability for a class of nonlinear parabolic equations, Nonlinear Anal. 7 (1994), 911–924
142. Q. Zhang, *A new critical phenomenon for semilinear parabolic problems*, J. Math. Anal. Appl. 219 (1998), 125–139
143. Q. Zhang, *Blow up and global existence of solutions to an inhomogeneous parabolic system*, J. Differ. Equat. 147 (1998), 155–183
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145. S.N. Zheng, *Nonexistence of positive solutions to semilinear elliptic system and blow-up estimates for reaction-diffusion system*, J. Math. Anal. Appl. 232 (1999), 293–311

146. S.N. Zheng, *Global existence and global nonexistence of solutions to a reaction-diffusion system*, Nonlinear Anal. 39 (2000), 327–340
147. S.N. Zheng, X.F. Song and Z.X. Jiang, *Critical Fujita exponents for degenerate parabolic equations coupled via nonlinear boundary flux*, J. Math. Anal. Appl. 298 (2004), 308–324
148. T.I. Zelenyak, *On quaitative properties of solutions of quasilinear mixed problems for equations of parabolic type*, (in Russian) Mat. Sb. 104 (1977), 486–510

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Edited by J.-M. Morel, B. Teissier, P.K. Maini

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